THE THROUGHPUT RATE
OF
MULTISTATION PRODUCTION LINES

BY

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To

my mother,

my sisters, my brother,

my friends

(In Indonesia)

"They also serve who only [pray] and wait"

adapted from John Milton ("On His Blindness")

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"Coincidences, in general, are great stumbling-blocks in the way of that class of thinkers who have been educated to know nothing of the theory of probabilities: that theory to which the most glorious objects of human research are indebted for the most glorious of illustration."
- Edgar Allan Poe (The Murders in the Rue Morgue)

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To these four fine scholars I give credit for most of the good parts of this dissertation. The rest is reserved to myself.

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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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A holding time model of serial production lines with an arbitrary number of stations and an arbitrary distribution of service times is presented. A system of integral equations for the distribution of interdeparture time is derived. For a special class of distributions of service times, two numerical procedures, one exact and the other approximate, for calculating the line throughput rate are given. For the three-station case the exact procedure is specialized into a formula for the throughput rate of a three-station production line. This formula is used to investigate the effect of line unbalance on the throughput rate. A model of production lines with station breakdown and repair is also considered and a method to obtain an equivalent production line without station breakdown is derived.

CHAPTER 1

INTRODUCTION

1.1. General Description of a Production Line

The production line is a common material handling and processing device of modern manufacturing systems, and the design of efficient production lines is one of the most important jobs of industrial engineers.

The production line model to be considered here is a series arrangement of work stations. At each station a specific operation is performed on the workpiece passing through the station. Each workpiece passes through all of the stations in the same sequence. An important measure of the line's efficiency is its throughput rate or mean production rate which is defined to be the number of workpieces released from the line per unit time in the long run.

An ideal production line is one in which

- the station service time is deterministic
- the line is balanced, which means that every station processes the same number of items per unit time
- there is no station breakdown.

Actual production lines suffer losses in efficiency owing to three distinct causes. These causes are

- randomness of the station service times
- unbalance of the line
- station breakdown.

Each of the causes results in a reduction in the line efficiency. Losses in efficiency, as observed on a particular station, are evidenced by periods where that station is either broken down, blocked or starved. A station may break down and cease to operate because of a direct station failure, or it may have to be shut down because the performance of the station is not within the bounds of prescribed tolerances. A station is blocked if it has completed service but cannot pass on its workpiece to the next station because that station is busy, blocked or broken down. A station is starved if it has passed on its workpiece to the next station but has not yet received a new workpiece from the preceding station. Blocking and starving are caused by unbalance, randomness of station service times and station breakdown. The effect of blocking propagates backward through the line, while the effect of starving propagates forward. A station breakdown will cause preceding stations to be blocked and succeeding stations to be starved. Thus there is a strong coupling effect between stations.

The coupling effect can be reduced by placing buffers, or in-process storage, between stations. Buffers have a decoupling effect since an otherwise starved station can still be fed from the preceding buffer, as long as this

buffer is not empty. Similarly, a blocked station can release its finished workpiece into the subsequent buffer, as long as this buffer is not full. A finite number of buffer does not completely decouple adjacent stations. Station interference still exists when the buffer is either empty or full. In practice buffers are of small size. A buffer with a capacity for L items is equivalent to L stations with zero service times in series. Therefore, in principle, the case of a line with buffers does not need to be treated differently from that of a line without buffers.

The throughput rate is a function of

- the number of stations
- the distribution of station service times
- the distribution of station repair times
- the distribution of station uptime between breakdowns

In this dissertation we investigate models that incorporate these functional relationships and we develop methods for computing the throughput rate.

1.2. Literature Review

Research on production lines appeared in the literature in the early 1950s. According to Buzacott [1972], production lines were first studied analytically via a probabilistic approach by Vladzievskii [1952].

One common probabilistic model for studying production lines is the system state model. The various states that the system may assume and the transition characteristics

among them are identified. Determination of throughput rate is based on finding the steady state probabilities for those states in which the last station is actively processing workpieces.

Simulation models have been applied with the expectation of obtaining some of the properties of certain production lines. It seems unlikely though that simulation will uncover general relationships. Nevertheless, simulation can be considered as a useful tool for verifying and comparing different models.

Continuous models have also been used to represent production lines. In these models workpieces are treated as a continuous fluid rather than as discrete items.

Individual studies differ by the model used to represent a production line and by the method of analysis. One major difference is whether or not station breakdown is considered. The case of station breakdown is further differentiated by considering the station service time to be either random or fixed. We divide the literature into three major classes, according to the model used. The first class consists of papers that treat models of production lines in which station breakdown is not considered. The second class consists of papers that treat models of production lines in which stations break down at random and station service times are fixed and identical. The third class consists of papers that treat models of production lines in which stations break down at random and station service times are

random variables. The literature review that follows is divided into these three classes. We only discuss papers that use analytical methods that are based on discrete models of production lines, since these are related to the model and methods of analysis of this dissertation. Simulation models can be found in Freeman [1964], Anderson and Moodie [1969], Masso and Smith [1974] and Ho et al. [1979]. Continuous models can be found in Sevast'yanov [1962], Murphy [1978], Wijngaard [1979] and Gershwin and Schick [1980].

Unless otherwise stated, all papers reviewed here assume that there is an infinite supply of workpieces to the first station and that there is always space available into which the last station can discharge a finished item.

1.2.1. Models of Production Lines without Station Breakdown

Hunt [1956] considers production lines with exponential service times. Using a Markov model he derives a simple closed form expression for the throughput rate of a two-station balanced production line with an arbitrary number of buffers between stations. He also derives a closed form expression for the throughput rate of a three-station production line with no buffers. Even though the throughput rate is a function of only three variables, the expression turns out to be very complicated. It has a numerator polynomial of degree 8 with 22 terms and a denominator polynomial of degree 7 with 24 terms. This complicated

formula is an indication that it is practically impossible to obtain closed form expressions for the throughput rate of a production line with more than three stations. Hunt also gives a numerical solution for a four-station production line with no buffers and a three-station balanced production line with buffers of capacity one.

Avi-Itzak and Yadin [1965] study a two-station production line with general service times and with no buffers. Items arrive at this production line according to a Poisson process. The authors obtain the moment generating function for the random variable time spent in the line by an item.

Hillier and Boling [1966], using the result of Hunt [1956], show numerically that the throughput rate of a three-station production line with exponential service time can be increased by assigning a lower mean service times to the middle station than to the two outside stations. They refer to this allocation of mean service times as "bowl phenomenon."

Hillier and Boling [1967], using a Markov model, develop a numerical procedure for calculating the line throughput rate. They treat lines in which all stations are identical and have the same number of buffers, for the case of exponential and Erlang service times. Their solution method exploits the sparsity of the associated transition rate matrix. Their numerical results are limited to at most six stations and at most four buffers per station. They also give an approximation method for calculating the

throughput rate of a production line with exponential service times. This method is based on Burke's theorem.

Muth [1973] considers a production line with an arbitrary distribution of service times and an arbitrary number of stations. He gives methods for calculating an upper and a lower bounds on the throughput rate. He also derives an integral equation for the distribution of the interdemand times of a three-station production line.

Rao [1975a] uses the model introduced by Muth [1973] to calculate the throughput rate of two-station production lines with Erlang, uniform and normal service times. In a second paper, Rao [1975b] uses his previous result to develop a method for calculating throughput rates of two-station production lines with an arbitrary number of buffers, where the first station has exponential service times and the second station has Erlang, uniform or normal service times.

Using the integral formulation for a three-station production line given by Muth [1973], Rao [1976] derives closed form expressions for throughput rates of several models of production lines. In the first model two stations have exponential service times and one has a deterministic service time, in the second model two stations have deterministic service times and one has an exponential service time, and in the third model the two outside stations have exponential service times and the middle station has a uniform service time. Using these closed form expressions,

he shows numerically that the bowl phenomenon does not always occur in a production line where stations have widely different coefficients of variation of service times. He finds that the throughput rate can be increased by assigning a lower mean service times to stations with higher variabilities. He calls this allocation "variability imbalance." He concludes that for some cases the effect of variability imbalance is opposite to the effect of mean value unbalance (bowl phenomenon of Hillier and Boling [1966]) and that the two effects may cancel one another.

Muth [1977], as in his earlier paper, considers a production line with an arbitrary distribution of service times and an arbitrary number of stations. He uses a passage time model to derive relations among the random variables involved. For the three-station case he derives a system of integral equations for the distribution of the interdemand times. He also gives two numerical procedures for solving this system of integral equations in the case of identical uniform service times and identical Erlang service times.

Hillier and Boling [1979], using the numerical procedure they developed in their earlier paper [1967], describe the bowl phenomenon for symmetrical production lines with up to six stations and coefficient of variation of service times equal to $1/\sqrt{m}$ where $m = 1, 2, . . ., 7$. They assume that the service times of all stations are Erlang with the same coefficient of variation. They show

that in all cases considered, the line throughput rate can be increased by assigning mean service times that decrease toward the middle of the line.

Altiok [1982] proposes an approximation method for calculating the throughput rate of a production line where all stations have exponential service times. His method is based on the no-memory property of exponential random variables and the assumption that two consecutive stations are never blocked at the same time. His method, as with Hillier and Boling's approximation method, gives a good result only when there are buffers of large size placed between stations.

Muth [1984] uses a passage time model to give activity network representations of a production line with an arbitrary distribution of service times and an arbitrary number of stations. These networks are a useful tool for deriving relationships among random variable involved. He gives a system of integral equations for the distribution of the interdemand times of a three-station production line. In the case all stations have exponential service times, these equations are reduced into a simple closed form expression compared to Hunt's. He also derives a numerical procedure for the case of two-stage general Erlang service times.

1.2.2. Models of Production Lines with Station Breakdown and Fixed Service Times

Buzacott [1967] derives an exact formula for the throughput rate of a two-station production line with finite

buffers. For a three-station line with identical repair time, an iterative solution procedure for finding an approximate throughput rate is given. The extension of this iterative procedure to lines with more than three stations is discussed. The iterative approximation is based on a process of dividing the production line into two stages and applying the formula derived for the two-station line. He also gives some insight on how and where to divide the line into stages. In a second paper, Buzacott [1968] considers several line configurations where stations are arranged in any combination of parallel-series and series-parallel forms and where there are either no buffers or there is an infinite number of buffers.

Sheskin [1976] assumes geometrically distributed repair times and time between breakdowns. These assumptions give rise to a discrete parameter Markov process. He also assumes that starved or blocked stations can break down and $q_i + r_i = 1$, where q_i is the conditional probability that station i breaks down at cycle n + 1 given that it was operational during cycle n, and r_i is the conditional probability that station i is repaired during cycle n + 1 given that it was under repair during cycle n. Exact numerical solutions for the steady state probabilities of the resulting Markov chain are obtained by a method he calls "compression" of the transition probability matrix. Approximate numerical solutions are obtained by a method he

calls "decomposition." Using these methods, he is able to calculate the throughput rate of a four-station production line containing up to twelve buffers.

Okamura and Yamashima [1977] assume geometrically distributed repair times and time between breakdowns. They compute the throughput rate of a two-station production line by solving a system of (4M + 6) simultaneous equations, where M is the number of buffers. Their numerical procedure, unlike the approach of Hillier and Boling [1967], does not exploit the structure of the transition probability matrix.

Ignall and Silver [1977] base their approach upon results obtained by Buzacott [1967, 1968] for the limiting cases of a two-station production line with no buffers and with an infinite number of buffers. An approximation formula for a two-station production line is developed such that the throughput rate is a linear combination of the throughput rates of the two limiting cases just mentioned.

Muth and Yeralan [1981] discover a particular Markov model of a two-station production line with an arbitrary number of buffers whose steady-state probabilities possess the scalar-geometric property. Using this property, these probabilities, and therefore the throughput rate, can be computed easily. Yeralan [1983] also shows that the steady-state probabilities of a Markov model of any two-station production line with an arbitrary number of buffers possess

the matrix-geometric property provided the states are ordered in a certain way.

Gershwin and Schick [1983] make the same assumption as Okamura and Yamashima. Using a Markov model they calculate the steady-state probabilities by assuming that these probabilities are in a sum of products form. Their method is restricted to two- and three-station production lines.

Shantikumar and Tien [1983] present a Markov model of a two-station production line which includes the possibility of scrapping an item in a broken down station. Using a concept called "level crossing analysis," a recursive solution procedure for the steady-state probabilities is given.

1.2.3. Models of Production Lines with Station Breakdown and Random Service Times

Buzacott [1972] considers a balanced two-station production line. He assumes that service times, repair times and time between breakdowns are exponentially distributed. These assumptions give rise to a continuous parameter Markov process, which is solved for the mean interdeparture time of items at station 2. Buzacott compares exact results obtained using this method with results obtained using a heuristic procedure for two-station production lines, and extrapolates this procedure to production lines with more than two stations.

Muth [1979a] studies general properties of production lines subject to breakdown without making a specific assumption on the distributions of service times, repair times or time between breakdowns. Several models for the line availability using different failure mechanism are given. Availability is defined as the ratio of the throughput rate of a production line with station breakdown to the throughput rate of the same production line without station breakdown. Muth also shows that the effects of service time variability and station breakdown can be combined under certain assumptions. This means that a production line subject to breakdown can be represented by an equivalent production line not subject to breakdown.

Gershwin and Berman [1981] present the same Markov model of a two-station production line as Buzacott [1967]. They calculate the steady-state probabilities by assuming that these probabilities are in a sum of products form. They find that these probabilities can be obtained by solving a system of nonlinear equations. The number of these equations is always five irrespective of the number of buffers.

Berman [1982] uses the assumption that service on an item starts all over after a station repair. Through this assumption he can extend the result of Gershwin and Berman [1981] to the case of Erlang service times.

Altiok and Stidham [1983] incorporate the breakdown-repair phenomenon into the service time distribution as in

Muth [1979a]. They assume that service times, repair times and time between breakdowns are exponential. Using a Markov model, they calculate the steady-state probabilities of a three-station production line.

1.3. Current Work

Even though there has been a lot of research conducted in the production line area, there are two major features that set this dissertation aside from previous research:

- The analysis is carried out for an arbitrary number of stations and for arbitrary distribution functions of service times, repair times and time between breakdowns at a station.
- The numerical procedures developed are for a general class of distribution functions.

This dissertation is organized as follows. In Chapter 2 we deal with a production line model in which station breakdown is not considered. For a special class of service time distributions that we call special phase type distributions, we develop two numerical procedures, one exact and the other approximate, for calculating the line throughput rate. We use these numerical procedures to investigate the effect of the number of stations and of the service time variabilities on the throughput rate of a balanced production line.

In Chapter 3 we incorporate station breakdown and repair in the model of Chapter 2. We derive a method to

obtain an equivalent production line without breakdown and we state conditions under which the solution procedures of the previous chapter can still be used.

In Chapter 4, using the method of Chapter 2, we investigate the effect of line unbalance on the throughput rate. We express the throughput rate as a function of mean value unbalance and variability unbalance. We present the results in the form of contours of constant throughput rate.

In Appendix A we give several properties of the class of special phase type distributions that are important for the development of solution procedures of Chapter 2. In Appendix B we specialize the method of Chapter 2 to a formula for the throughput rate of a three-station production line. We use this formula in Chapter 4 to investigate the effect of line unbalance on the throughput rate.

CHAPTER 2

PRODUCTION LINES WITHOUT STATION BREAKDOWN

2.1. Introduction

In the past production lines with stochastic servers have typically been modeled as discrete state Markov processes. In such models, the service time of the work stations is exponentially distributed. Extensions to Erlang service times are feasible at the expense of a considerable growth in the numbers of system states. In Muth [1977, 1984] a holding time model is presented that is not based on the Markov property and is thus, in principle, not restricted to any particular type of service time distribution. Furthermore, that model does not suffer from the explosive growth of the state space as experienced by the Markov model. The holding time model leads to an integral equation defining the distribution of the interdemand time, and thus defining the system throughput rate. Closed form solutions for the production rate when there are up to three work stations, and for special distributions, are given in Muth [1984]. Numerical solutions for up to three work stations and more general distributions present no problem. In this chapter we extend the result of Muth [1984] by developing a solution for the case of more than

three work stations. We present two numerical procedures, one exact the other approximate, for solving the integral equations when the service time distributions are of special phase type. The approximate method gives a very tight lower bound to the actual production rate. It is of interest because it is less complex than the exact procedure and because it can handle a larger class of service time distributions.

The chapter is organized as follows. In section 2.2 we give the general assumptions pertaining to a system characterized by its states and in section 2.3 we discuss the corresponding holding time model. Subsequently we establish a system of integral equations for the interdemand time, and we present an exact numerical solution method for the case of exponentially distributed station service times. This is followed by an approximate numerical method for the same case. Finally, in section 2.7 we generalize the numerical solution methods to the case of special phase type station service times.

2.2. Model Assumptions

We are treating a general model of a production line, consisting of K work stations (servers) labeled 1, 2, ..., K, and arranged in series in that order. There is an unlimited supply of workpieces, called production items, at station 1. Each item enters the line at station 1, passes through all stations in order, and leaves station K in

finished form. At each station a service is performed. The service time of item n at station j is denoted $S_j(n)$. The sequence $S_j(n)$ is a sequence of independent nonnegative random variables, identically distributed as S_j. The random variables $S_1, S_2, \ldots, S_K$ are independent too; their distributions are not necessarily identical. It is assumed that a station cannot break down and that each station can service only one item at a time. Every station is at any time in one of three possible states. A station is busy when it is servicing an item, it is blocked when the item on which service has been completed cannot move on to the subsequent station because that station is busy or blocked, and it is starved or empty otherwise. The starved state is caused when an item has left the station, and service on the item in the preceding station has not yet been completed. The durations of these states associated with the passage of a single item are called busy period, blocking period and starving period. Service begins immediately when an item moves into a station. Station 1 can never be starved and station K can never be blocked. There may be buffer spaces (in-process storage) installed between stations to diminish the occurrence of blocking and starving. This case does not require special treatment since a buffer with a capacity of L items is equivalent to L stations with zero service time in series.

With these assumptions the production line is completely specified given the number K and the distribution function $F_{S_j}(t)$ of the station service time S_j.

2.3. The Holding Time Model

We give a brief review of the main features of the holding time model presented in Muth [1984]. The model is based on a stochastic process whose transitions occur when an item moves from one station to the next. A move of an item from station j to station j+1 corresponds to an arrival at station j+1. For $j \in \kappa$, $\kappa = \{1, 2, ..., K\}$ we define the random variables

$B_j(n)$ = blocking period of item n at station j

$H_j(n)$ = period spent by item n in station j, or holding period of item n at station j

$I_j(n)$ = starving period of station j following the departure of item n-1

$S_j(n)$ = service period of item n at station j

and the random vectors

$H(n) = [H_1(n), H_2(n), . . ., H_K(n)]$

$I(n) = [I_1(n), I_2(n), . . ., I_K(n)]$

$S(n) = [S_1(n), S_2(n), . . ., S_K(n)]$

From these definitions, we have

$$H_j(n) = S_j(n) + B_j(n) \tag{2-3-1}$$

The service period process $\{S(n): n\varepsilon N\}$ is a superposition of K independent and nonidentical renewal processes. It induces the holding period process $\{H(n): n\varepsilon N\}$ and the starving period process $\{I(n): n\varepsilon N\}$. The following recursive relationships were derived in Muth [1984]:

$$H_j(n) = \max\ [S_j(n),\ H_{j+1}(n-1) - I_j(n)] \text{ for } j\varepsilon\kappa \qquad (2\text{-}3\text{-}2)$$

$$I_j(n) = \max\ [I_{j-1}(n) + S_{j-1}(n) - H_j(n-1),\ 0] \text{ for } j\varepsilon\kappa \qquad (2\text{-}3\text{-}3)$$

To make (2-3-2) and (2-3-3) hold in all cases, $H_j(n)$, $I_j(n)$ and $S_j(n)$ are defined to be identically zero for $j = 0$ and $j = K+1$. We also have

$$H_K(n) = S_K(n) \qquad (2\text{-}3\text{-}4)$$

$$I_1(n) = 0 \qquad (2\text{-}3\text{-}5)$$

The system of recursive equations (2-3-2) to (2-3-5) defines how H(n) and I(n) are induced by S(n). Of special interest in (2-3-2) and (2-3-3) is the random variables

$$R_j(n) = \begin{cases} H_j(n) - I_{j-1}(n+1) & \text{for } j\varepsilon\kappa \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}3\text{-}6)$$

which represents the time difference between the departure of item n from station j and the arrival of item n+1 at station j-1. This random variable can be either negative or positive. We therefore define

$$R_j^+(n) = \max\ [R_j(n),\ 0] \tag{2-3-6a}$$

$$R_j^-(n) = \min\ [R_j(n),\ 0] \tag{2-3-6b}$$

The random variable $R_j^+(n)$ is the residual holding period of item n at station j following the entrance of item n+1 into station j-1. The absolute value of $R_j^-(n)$ is the time period during which both stations j and j-1 are starved together. Substituting (2-3-6) into (2-3-2) and (2-3-3) we obtain for all $j \varepsilon \kappa$

$$H_j(n) = \max\ [S_j(n),\ R_{j+1}^+(n-1)] \tag{2-3-7}$$

$$I_j(n) = \max\ [S_{j-1}(n) - R_j(n-1),\ 0]. \tag{2-3-8}$$

Equations (2-3-4) through (2-3-8) serve as the basic equations used to express a relationship between the distribution functions of service period, holding period and starving period.

The relationship among the above variables can be nicely shown through activity networks as is done in Muth

[1984]. The networks are a useful tool for deriving such equations as are used in this section.

2.4. Integral Equations for the Holding Time Model

If all service times have finite mean values, then it is plausible that $H_j(n)$, $I_j(n)$, $R_j^+(n)$ and $R_j^-(n)$ converge in distribution to random variables H_j, I_j, R_j^+ and R_j^- as $n \to \infty$. For a discussion see Muth [1984]. Thus from equations (2-3-6) through (2-3-8) the recursive equations for the distribution function F of the above quantities can be obtained.

If $H_j(n-1)$ is statistically independent of $I_{j-1}(n)$ then from equations (2-3-6) and (2-3-6a), for all $j \varepsilon \kappa$ we obtain

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x)\, dF_{H_j}(x+t) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}4\text{-}1)$$

and from equation (2-3-6) and (2-3-6b), for all $j \varepsilon \kappa$ we obtain

$$F_{R_j^-}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t)\, dF_{H_j}(x) & \text{for } t \leq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}4\text{-}2)$$

If $R_{j+1}(n-1)$ is statistically independent of $S_j(n)$, then from equations (2-3-7) and (2-3-8), for all $j\epsilon\kappa$ we obtain

$$F_{H_j}(t) = F_{S_j}(t)F_{R^+_{j+1}}(t) \tag{2-4-3}$$

$$F_{I_j}(t) = \begin{cases} 1 - \int_{x=-t}^{0} F_{R_j^-}(x)\,dF_{S_{j-1}}(x+t) - \int_{x=0}^{\infty} F_{R_j^+}(x)\,dF_{S_{j-1}}(x+t) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2-4-4}$$

(see Appendix C for the derivation of these equations).

Equations (2-4-1) through (2-4-4) are the system of equations defining $F_{H_j}(t)$, $F_{I_j}(t)$, $F_{R_j^+}(t)$ and $F_{R_j^-}(t)$ for all $j\epsilon\kappa$. For $j = 0$ and $j = K+1$ they are equal to $u(t)$ which is the unit step function with jump point at $t = 0$.

It follows from the assumed statistical independence of $S_{j+1}(n-1)$ and $S_j(n)$ that $R_{j+1}(n)$ and $S_j(n)$ are also statistically independent, which can be seen in the network representation given in Muth [1984]. The question of whether or not $H_j(n-1)$ and $I_{j-1}(n)$ are statistically independent when $K > 3$ is a difficult one. We have not been able to resolve this question from an analysis of the complex relationships. Therefore we have conducted extensive simulations of four- and five-station balanced production lines with exponential service times. As a result of

100 simulation runs, the 95% confidence interval for the sample correlation coefficient of $H_j(n-1)$ and $I_{j-1}(n)$ is [0.003, 0.006]. This result indicates a very slight positive correlation. For practical purposes, however, we may conclude that $H_j(n-1)$ and $I_{j-1}(n)$ are statistically independent.

Even when the index j represents a station and the index j-1 represents a buffer, or vice versa, the sample correlation coefficient of $H_j(n-1)$ and $I_{j-1}(n)$ is about the same as above. However, when the indices j-1 and j are both for buffers, the sample correlation coefficient of $H_j(n-1)$ and $I_{j-1}(n)$ is much higher. For two-station balanced production lines with a buffer of capacity two or three and exponential service times, the 95% confidence interval for the sample correlation coefficient of $H_j(n-1)$ and $I_{j-1}(n)$ is [0.03, 0.09].

Substitution of (2-4-3) into (2-4-1) and (2-4-2) gives, for all $j \varepsilon \kappa$,

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x)\, d(F_{S_j}(x+t) F_{R_{j+1}^+}(x+t)) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2\text{-}4\text{-}5)$$

$$F_{R_j^-}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t)\, d(F_{S_j}(x) F_{R_{j+1}^+}(x)) & \text{for } t \leq 0 \\ 1 & \text{otherwise} \end{cases} \quad (2\text{-}4\text{-}6)$$

Given the set of $F_{I_j}(t)$ and $F_{S_j}(t)$ for all $j \varepsilon \kappa$, the set of $F_{R_j^+}(t)$ can be generated using equation (2-4-5) and backward substitution starting with $F_{R_{K+1}}(t) = u(t)$. Likewise given the set of $F_{R_j^+}(t)$ and $F_{S_j}(t)$ for all $j \varepsilon \kappa$, the set of $F_{R_j^-}(t)$ and $F_{I_j}(t)$ can be generated alternately using equation (2-4-4) and (2-4-6) and forward substitution starting with $F_{I_1}(t) = u(t)$ or $F_{R_2^-}(t) = u(t)$ (since $I_1 \equiv 0$, $R_2 = H_2$ which is always positive). Thus we have three functional relationships between the set of $F_{R_j^+}(t)$, the set $F_{R_j^-}(t)$ and the set of $F_{I_j}(t)$. In combination these three relationships establish a system of simultaneous equations defining $F_{R_j^+}(t)$, $F_{R_j^-}(t)$ and $F_{I_j}(t)$.

2.5. The System Throughput Rate

The throughput rate r of the production line is defined as the number of items entering the first station per unit time, in the long run. Let N(t) be the number of items entering the first station in the time interval (0,t). Then

$$r = \lim_{t \to \infty} \frac{E[N(t)]}{t} \qquad (2\text{-}5\text{-}1)$$

This is equivalent to

$$r = \frac{1}{E[H_1]} \qquad (2\text{-}5\text{-}2)$$

where

$$E[H_1] = \int_0^\infty (1-F_{H_1}(t))dt \qquad (2\text{-}5\text{-}3)$$

The random variable H_1 is the interdemand time of station 1. Since the first station is never starved, the interarrival time of items to the production line is distributed as H_1. By the assumption that $H_j(n)$ converges in distribution to H_j, H_1 has a finite mean. The equivalence between (2-5-1) and (2-5-2) thus follows from the law of large number (see Cox [1962]). From (2-4-3) it follows that

$$F_{H_1}(t) = F_{S_1}(t)F_{R_2^+}(t) \qquad (2\text{-}5\text{-}4)$$

Since the first station is never starved it is seen from (2-3-6) that $R_2^+ = H_2$; hence using (2-4-3) again we have

$$F_{H_1}(t) = F_{S_1}(t)F_{S_2}(t)F_{R_3^+}(t) \qquad (2\text{-}5\text{-}5)$$

To compute $F_{H_1}(t)$ we determine $F_{R_3^+}(t)$ from the simultaneous equations established by (2-4-4) through (2-4-6). A solution procedure for the case where S_j for all $j\varepsilon\kappa$ are exponentially distributed is presented in the following section.

2.6. Solution for Exponential Service Times

We are treating the case where all stations have exponential service times, i.e.

$$F_{S_j}(t) = \begin{cases} 1-e^{-\alpha_j t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}6\text{-}1)$$

For this choice of service time distribution $F_{R_{j+1}^+}(t)$ takes the following form (see appendix A for a formal proof):

$$F_{R_{j+1}^+}(t) = \begin{cases} 1 - \sum_{i=1}^{m_j} a_{ij} e^{-s_{ij} t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}6\text{-}2)$$

where $s_{ij} > 0$ and $0 \leq \sum_{i=1}^{m_j} a_{ij} \leq 1$.

Equation (2-6-2) can be written as

$$F_{R_{j+1}^+}(t) = \sum_{i=0}^{m_j} c_{ij} e^{-s_{ij} t} \qquad \text{for } t \geq 0 \qquad (2\text{-}6\text{-}3)$$

where $c_{0j} = 1$, $s_{0j} = 0$ and $c_{ij} = -a_{ij}$ for $i \neq 0$.

Substitution of (2-6-3) into (2-4-5) gives

$$F_{R_j^+}(t) = 1 + \sum_{i=0}^{m_j} c_{ij} G(s_{ij}, j, t) \qquad (2\text{-}6\text{-}4)$$

where

$$G(s_{ij},j,t) = -\int_{x=0}^{\infty} F_{I_{j-1}}(x)\, d(F_{S_j}(x+t)e^{-s_{ij}(x+t)})$$

$$= A(j-1,s_{ij})e^{-s_{ij}t} - A(j-1,s_{ij}+\alpha_j)e^{-(s_{ij}+\alpha_j)t} \tag{2-6-5}$$

and

$$A(j,s) = \begin{cases} \int_{x=0}^{\infty} F_{I_j}(x)\, s e^{-sx} dx & \text{for } s \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2-6-6}$$

Equations (2-6-2) through (2-6-6) are the recursive equations that can be used to generate $F_{R_j^+}(t)$ from $F_{R_{j+1}^+}(t)$. However, one problem with these equations is that the parameters a_{ij} and s_{ij} are still unknown. Therefore we will bring the equations into a form which does not contain a_{ij} and s_{ij}. Toward this end we define a linear operator * as follows:

$$F_{S_j}(t) * e^{-st} = A(j-1,s)e^{-st} - A(j-1,s+\alpha_j)e^{-(s+\alpha_j)t} \tag{2-6-7}$$

By comparing the right hand sides of (2-6-5) and (2-6-7), and noting the form of $F_{R^+_{j+1}}(t)$ it is easy to see that equation (2-6-4) can be rewritten as

$$F_{R^+_j}(t) = 1 + F_{S_j}(t) * F_{R^+_{j+1}}(t) \tag{2-6-8}$$

Equation (2-6-8) represents a method that recursively generates $F_{R^+_j}(t)$ from $F_{R^+_{j+1}}(t)$ starting with $F_{R^+_{K+1}}(t) = e^{0t} \equiv 1$. For j=K and j=K-1, the explicit form of $F_{R^+_j}(t)$ is shown below.

$$\begin{aligned} F_{R^+_K}(t) &= 1 + F_{S_K}(t) * e^{0t} \\ &= 1 + A(K-1,0)e^{0t} - A(K-1,\alpha_K)e^{-\alpha_K t} \\ &= 1 - A(K-1,\alpha_K)e^{-\alpha_K t} \end{aligned}$$

$$\begin{aligned} F_{R^+_{K-1}}(t) &= 1 + F_{S_{K-1}}(t) * (1-A(K-1,\alpha_K)e^{-\alpha_K t}) \\ &= 1 - A(K-2,\alpha_{K-1})e^{-\alpha_{K-1} t} - A(K-1,\alpha_K) \times \\ &\qquad (A(K-2,\alpha_K)e^{-\alpha_K t} - A(K-2,\alpha_K+\alpha_{K-1})e^{-(\alpha_K+\alpha_{K-1})t}) \end{aligned}$$

To make the procedure useful for numerical calculation, define a function $G_k(j,s)$ as follows:

$$G_0(j,s) = e^{-st}$$

$$G_1(j,s) = F_{S_j}(t) * e^{-st}$$

$$G_2(j,s) = F_{S_j}(t) * F_{S_{j+1}}(t) * e^{-st} = F_{S_j}(t) * G_1(j+1,s)$$

$$G_k(j,s) = F_{S_j}(t) * F_{S_{j+1}}(t) * \ldots * F_{S_{j+k-1}}(t) * e^{-st}$$

(2-6-9)

$G_k(j,s)$ is a superposition of exponential functions. The number of exponential terms in $G_k(j,s)$ is 2^k. From equations (2-6-7) and (2-6-9) we can see that $G_k(j,s)$ satisfies the following recursion:

$$G_0(j,s) = e^{-st}$$

$$G_k(j,s) = A(j+k-2,s)G_{k-1}(j,s)$$

$$- A(j+k-2,s+\alpha_{j+k-1})\ G_{k-1}(j,s+\alpha_{j+k-1})$$

for $k = 1, 2, \ldots$ (2-6-10)

Using the factor c_i defined below and applying equations (2-6-7) through (2-6-9), the evolution of $F_{R_j^+}(t)$ is shown for several values of j as follows:

$$
\begin{array}{ll}
j & F_{R_j^+}(t) \\
\\
K+1 & e^{0t} \\
\\
K & e^{0t} + c_K e^{-\alpha_K t} \\
\\
K-1 & e^{0t} + c_{K-1} e^{-\alpha_{K-1} t} + c_K G_1(K-1, \alpha_K) \\
\\
K-2 & e^{0t} + c_{K-2} e^{-\alpha_{K-2} t} + c_{K-1} G_1(K-2, \alpha_{K-1}) + c_K G_2(K-2, \alpha_K)
\end{array}
$$

and so on.

Thus, noting that $G_0(j,s) = e^{-st}$ and $e^{0t} \equiv 1$, we have that

$$F_{R_j^+}(t) = 1 + \sum_{i=j}^{K} c_i G_{i-j}(j, \alpha_i) \tag{2-6-11}$$

where, from equation (2-6-7) with s = 0,

$$c_i = -A(i-1, \alpha_i)$$

Hence

$$F_{R_j^+}(t) = 1 - \sum_{i=j}^{K} A(i-1,\alpha_i)G_{i-j}(j,\alpha_i) \qquad (2\text{-}6\text{-}12)$$

and we have the means to compute $G_k(j,s)$ readily at hand given by equation (2-6-10). As an illustration, we give the expressions of $F_{R_j^+}(t)$ for j=K and j=K-1 below.

$$F_{R_K^+}(t) = 1 - A(K-1,\alpha_K)G_0(K,\alpha_K)$$

$$= 1 - A(K-1,\alpha_K)e^{-\alpha_K t}$$

$$F_{R_{K-1}^+}(t) = 1 - A(K-2,\alpha_{K-1})G_0(K-1,\alpha_{K-1}) - A(K-1,\alpha_K)G_1(K-1,\alpha_K)$$

$$= 1 - A(K-2,\alpha_{K-1})e^{-\alpha_{K-1}t} - A(K-1,\alpha_K) \times$$

$$(A(K-2,\alpha_K)e^{-\alpha_K t} - A(K-2,\alpha_K+\alpha_{K-1})e^{-(\alpha_K+\alpha_{K-1})t})$$

As a next step, it is necessary to establish a recursive expression for $F_{H_j}(t)$ which corresponds to the expressions for $F_{R_{j+1}^+}(t)$ given in (2-4-3). From equation (2-4-3) and (2-6-12) we have

$$F_{H_j}(t) = F_{S_j}(t) - \sum_{i=j+1}^{K} A(i-1,\alpha_i)G'_{i-j-1}(j+1,\alpha_i)$$

(2-6-13)

where

$$G'_k(j+1,s) = G_k(j+1,s)F_{S_j}(t). \qquad (2\text{-}6\text{-}13a)$$

By comparing (2-6-13a) with the recursion (2-6-10) we see that $G'_k(j+1,s)$ satisfies that same recursion (2-6-10) given for $G_k(j+1,s)$ except that it starts with $G'_0(j+1,s) = e^{-st}F_{S_j}(t)$.

We define the parameter B_j as

$$B_j = \begin{cases} \int_0^\infty F_{R_j^+}(x)\alpha_{j-1}e^{-\alpha_{j-1}x}dx & \text{for } j=2,3,\ldots,K \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}6\text{-}14)$$

Substituting (2-6-12) into (2-6-14) gives

$$B_j = 1 - \sum_{i=j}^{K} A(i-1,\alpha_i)E_{i-j}(j,\alpha_i) \quad \text{for } j=2,3,\ldots,K$$

(2-6-15)

where

$$E_K(j,s) = \int_0^\infty G_k(j,s)\alpha_{j-1}e^{-\alpha_{j-1}x}dx;$$

$E_k(j,s)$ satisfies the recursion (2-6-10) given for $G_k(j,s)$ except that it starts with

$$E_0(j,s) = \int_0^\infty G_0(j,s)\alpha_{j-1}e^{-\alpha_{j-1}x}dx = \alpha_{j-1}/(s+\alpha_{j-1}).$$

At this point we have completed the derivation of recursive equations for $F_{R_j^+}(t)$ and $F_{H_j}(t)$. The next step is to derive similar equations for $F_{I_j}(t)$ and $F_{R_j^-}(t)$. However $F_{I_j}(t)$ and $F_{R_j^-}(t)$ are not of special phase type (see appendix A). It turns out that the form of $F_{I_j}(t)$ and $F_{R_j^-}(t)$ depends on the particular condition of the production line. An exact expression for $F_{I_j}(t)$ and $F_{R_j^-}(t)$ for the balanced case is obtained next. The unbalanced case has little practical interest.

2.6.1. The Exact Method: Balanced Line

The derivation so far allowed for the station service times to be nonidentical. From now on, in order to be able to keep the derivation of $F_{I_j}(t)$ and $F_{R_j^-}(t)$ simple, we require identical service times, and we therefore put

$\alpha_j = \alpha$. We alternately use equations (2-4-4) and (2-4-2) to derive $F_{I_j}(t)$ and $F_{R^-_{j+1}}(t)$, starting with j=2 and moving forward toward j=K-1. From $I_1 \equiv 0$, it follows that $R_2 = H_2$ so that R_2 is always positive. Hence from equations (2-4-4) and (2-6-14)

$$F_{I_2}(t) = 1 - B_2e^{-\alpha t}$$

and from (2-4-2)

$$F_{R^-_3}(t) = B_2Ce^{\alpha t}$$

where C is some constant. From equation (2-4-4) again

$$F_{I_3}(t) = 1 - B_3e^{-\alpha t} - \int_{x=-t}^{0} B_2Ce^{\alpha x}\alpha e^{-\alpha x}dxe^{-\alpha t}$$

$$= 1 - B_3e^{-\alpha t} - B_2C\alpha te^{-\alpha t}$$

Continuing in this fashion, we obtain

$$F_{I_j}(t) = 1 - \sum_{i=1}^{j-1} D_{ij}\frac{(\alpha t)^{i-1}}{(i-1)!}e^{-\alpha t} \tag{2-6-16}$$

Substitution of (2-6-16) into (2-4-2) gives

$$F_{R^-_{j+1}}(t) = \sum_{i=1}^{j-1} D_{ij}\left(\int_{x=0}^{\infty} \frac{\alpha^{i-1}}{(i-1)!}(x-t)^{i-1}e^{-\alpha x}dF_{H_{j+1}}(x)\right)e^{\alpha t}$$

$$= \sum_{i=1}^{j-1} D_{ij}\left(\sum_{k=0}^{i-1} \frac{\alpha^{i-1}(-1)^k}{k!\,(i-k-1)!}\int_{x=0}^{\infty} x^{i-k-1}e^{-\alpha x}dF_{H_{j+1}}(x)\right)t^k e^{\alpha t}$$

$$= \sum_{i=1}^{j-1} D_{ij} \sum_{k=0}^{i-1} \frac{(-\alpha)^k}{k!}\left(\int_{x=0}^{\infty} \frac{(\alpha x)^{i-k-1}}{(i-k-1)!} e^{-\alpha x}dF_{H_{j+1}}(x)\right)t^k e^{\alpha t}$$

$$= \sum_{i=1}^{j-1} D_{ij} \sum_{k=0}^{i-1} \frac{(-\alpha t)^k}{k!} C_{j,i-k}e^{\alpha t} \qquad (2\text{-}6\text{-}17)$$

where

$$C_{i,j} = \int_0^{\infty} \frac{(\alpha x)^{j-1}}{(j-1)!} e^{-\alpha x}dF_{H_{i+1}}(x) \quad \text{for } j < i \qquad (2\text{-}6\text{-}18)$$

and where $F_{H_j}(x)$ is given by (2-6-13). Substituting (2-6-13) into (2-6-18) we obtain

$$C_{ij} = \begin{cases} \dfrac{1}{2^j} + \displaystyle\sum_{k=i+2}^{K} A(k-1,\alpha)E'_{k-i-2}(i+2,j,\alpha) & \text{for } j < i \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}6\text{-}19)$$

where

$$E_k'(i,j,s) = \int_0^\infty \frac{(\alpha x)^{j-1}}{(j-1)!} e^{-\alpha x} dG_k'(i,s);$$

$E_k'(i,j,s)$ satisfies the same recursion as $G_k'(i,s)$ which in turn follows the same recursion as $G_k(i,s)$. The starting value is

$$E_0'(i,j,s) = \int_{x=0}^\infty \frac{(\alpha x)^{j-1}}{(j-1)!} e^{-\alpha x} dG_0'(i,s) = \frac{(s+\alpha)\alpha^{j-1}}{(s+2\alpha)^j} - \frac{s\alpha^{j-1}}{(s+\alpha)^j} .$$

In order to find $F_{I_{j+1}}(t)$ we evaluate the following integral:

$$\int_{-t}^0 F_{\bar{R}_{j+1}}(x)\alpha e^{-\alpha x} dx = \sum_{i=1}^{j-1} D_{ij} \sum_{k=0}^{i-1} \frac{(-\alpha)^k}{k!} C_{j,i-k} \int_{x=-t}^0 \alpha x^k dx$$

$$= \sum_{i=1}^{j-1} D_{ij} \sum_{k=0}^{i-1} C_{j,i-k} \frac{(\alpha t)^{k+1}}{(k+1)!}$$

Thus from (2-4-4) and (2-6-14) we obtain

$$F_{I_{j+1}}(t) = 1 - B_{j+1}e^{-\alpha t} - \sum_{i=1}^{j-1} D_{ij} \sum_{k=0}^{i-1} C_{j,i-k} \frac{(\alpha t)^{k+1}}{(k+1)!} e^{-\alpha t} \tag{2-6-20}$$

$$= 1 - \sum_{i=1}^{j} D_{i,j+1} \frac{(\alpha t)^{i-1}}{(i-1)!} e^{-\alpha t} \tag{2-6-21}$$

Comparing the coefficient of $\frac{(\alpha t)^{i-1}}{(i-1)!}\, e^{-\alpha t}$ in both (2-6-20) and (2-6-21), we have the following recursive equations for $D_{i,j+1}$, $i < j+1$:

$$D_{i,j+1} = \begin{cases} B_{j+1} & \text{for } i = 1 \\ \sum_{k=i-1}^{j-1} D_{k,j}\, C_{j,k-i+2} & \text{for } i = 2, 3, \ldots, j \end{cases} \tag{2-6-22}$$

Finally, substitution of (2-6-16) into (2-6-6) gives, for $s \neq 0$,

$$A(j,s) = \begin{cases} 1 - \sum_{i=1}^{j-1} D_{i,j} \frac{\alpha^{i-1} s}{(s+\alpha)^i} & \text{for } j = 2, \ldots, K \\ 1 & \text{otherwise} \end{cases} \tag{2-6-23}$$

Equations (2-6-15), (2-6-19), (2-6-22) and (2-6-23) constitute the system of equations that needs to be solved in order to find the value of D_{ij}, $i < j$, $j = 2,3,\ldots,K-1$ (D_{iK} is not needed). Once the D_{ij} have been determined, the system throughput rate can be calculated by substituting equation (2-6-12) for j=3 into (2-5-5) to obtain the mean interdemand time; that is

$$E[H_1] = \mu + \sum_{i=3}^{K} A(i-1,\alpha_i) E''_{1-3}(\alpha_i) \tag{2-6-24}$$

where

$$\mu = \int_0^\infty (1-F_{S_1}(t)F_{S_2}(t))dt \qquad (2\text{-}6\text{-}25)$$

$$E_k''(s) = \int_0^\infty G_k(3,s)F_{S_1}(t)F_{S_2}(t)dt; \qquad (2\text{-}6\text{-}26)$$

$E_k''(s)$ satisfies the recursion (2-6-10) given for $G_k(3,s)$ except that it starts with

$$E_0''(s) = \int_0^\infty e^{-st}F_{S_1}(t)F_{S_2}(t)dt.$$

The number of equations that need to be solved is in the order of K^2. The equations are nonlinear and are in the form of a fixed point problem, i.e. the problem of finding a vector x which satisfies x = f (x) where f is a vector valued function. We know that because of statistical equilibrium, B_j and C_{ij} are unique and all of their components must have values between 0 and 1. This information helps in choosing a good initial value for the solution procedure. A simple iterative scheme can be devised as follows:

Step 1: choose B_j^0 and C_{ij}^0; let k=0.

Step 2: calculate D_{ij}^k using (2-6-22)

calculate B_j^{k+1} using (2-6-15) and C_{ij}^{k+1} using (2-6-19)

Step 3: if $|B_j^{k+1} - B_j^k| < \varepsilon$ and $|C_{ij}^{k+1} - C_{ij}| < \varepsilon$ for all i and j, stop.

Otherwise let $k = k+1$ and return to step 2.

This method has been found to work well.

It is instructive to compare the method presented with that of solving a Markovian state model. Both involve the solution of a system of equations. The Markov model of a K station line has a total number of states which grows asymptotically as $(2.62)^K/2.24$. Thus the case K=6, which is the largest value of K for which a solution is given in the literature, results in 144 states (it grows to 6765 for K=10). This means that a system of 144 simultaneous equations must be solved to obtain the steady state probabilities. In the method of this paper there are $(K-2)(K-1)/2$ simultaneous equations (i.e. the number of variables B_j and C_{ij} that are needed to calculate the system throughput rate) to be solved. The convergence of the solution procedure given above has been found to be very fast. For the case K=10, it took 18 iterations with 30 seconds of run time on a VAX 11/750 under the VAX/VMS operating system. Throughput rates computed for values of K up to 10 are shown in Figure 2.1 where $r_N = E[S_1]/E[H_1]$, called the utilization or the normalized throughput rate, is graphed as a function of K. Comparing the results of this procedure and the result of Hillier and Boling [1967] shows that the difference in the case K=6 is about 0.1%. This

small difference could be due to the numerical accuracy of either method.

2.6.2. An Approximation Method

Since R_j^- represents the period of time where station j and j-1 are both starving, we can expect that R_j^- has a large probability of being zero, especially in a balanced line. Therefore, as an approximation, we may assume that R_j^- is zero with probability one. In terms of the defining equations this assumption affects only the expression for $I_j(n)$; i.e. equation (2-3-8) becomes

$$\hat{I}_j(n) = \max\,[S_{j-1}(n) - R_j^+(n-1), 0] \quad \text{for } j \in K \qquad (2\text{-}6\text{-}27)$$

Hence

$$F_{\hat{I}_j}(t) = 1 - \int_{x=0}^{\infty} F_{R_j^+}(x)\,dF_{S_{j-1}}(x+t)$$

$$= 1 - B_j\, e^{-\alpha_{j-1}t} \qquad \text{for } t \geq 0 \qquad (2\text{-}6\text{-}28)$$

Substitute (2-6-28) into (2-6-6) to obtain, for $s \neq 0$,

$$A(j,s) = \begin{cases} 1 - B_j s/(s+\alpha_{j-1}) & \text{for } j=2,3,\ldots,K \\ 1 & \text{otherwise} \end{cases} \qquad (2\text{-}6\text{-}29)$$

The solution of equations (2-6-29) and (2-6-15) yields the values of B_j, $j=2,\ldots,K-1$. Once the B_j are found the line throughput rate can be calculated using equation (2-6-24) where $A(j,s)$ is defined as in (2-6-29). In this approximation only (K-2) equations in the form of a fixed point problem need to be solved. Furthermore the equations themselves are much simpler. This reduction of complexity results in a considerable reduction in computational effort. For the case K=10, it took 8 iterations with 5 seconds of run time. Computed approximations are included in Figure 2.1.

An advantage of this approximation method is that it is applicable to a larger class of service time distributions. Moreover, the same solution procedure applies for identical and nonidentical server.

The approximate throughput rate is less than the actual throughput rate since $\hat{I}_j$ is never larger than I_j and therefore the corresponding H_j is never smaller than when it is calculated using I_j; see equation (2-3-2). Therefore, by using this approximation, $E[H_1]$ is larger than its actual value. It should be noted that any error introduced into the distribution $F_{\hat{I}_j}(t)$ using equation (2-6-28) will subsequently be diminished considerably when the corresponding $F_{R^+_{j+1}}(t)$ is used in equation (2-4-6). To explain the diminishing effect consider equation (2-3-7). The error present in the value of any particular realization of $\hat{I}_j(n)$ will become an error in the corresponding realization of $H_j(n)$

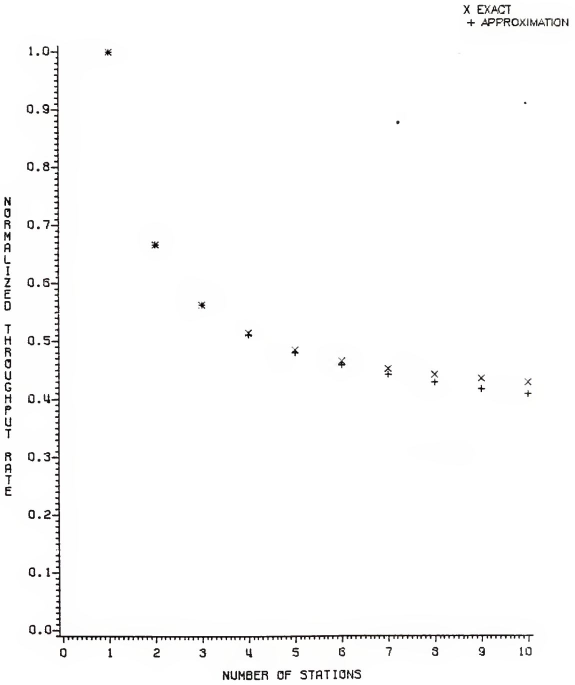


Figure 2.1. The normalized throughput rate as a function of the number of stations: exponential case.

only if $R_{j+1}(n-1) = H_{j+1}(n-1) - I_j(n) > S_j(n)$, since otherwise $H_j(n) = S_j(n)$. This condition will exist only for a small fraction of all realizations.

2.7. Solution for Special Phase Type Service Times

Let the service time distribution of the jth station be of special phase type discussed in Appendix A; i.e.

$$F_{S_j}(t) = 1 - \sum_{i=1}^{m_j} \beta_{i,j} e^{-\alpha_{i,j} t} \quad \text{for } t \geq 0. \tag{2-7-1}$$

Equation (2-6-8) still holds here, but with the operation * defined as follows:

$$F_{S_j}(t) * e^{-st} = A(j-1,s)e^{-st} - \sum_{i=1}^{m_j} A(j-1,\alpha_{i,j}+s)\beta_{i,j} e^{-(s+\alpha_{i,j})t} \tag{2-7-2}$$

where $A(j,s)$ is defined as in equation (2-6-6). From the definition of $G_k(j,s)$ as given in equation (2-6-9), the expression of $F_{R_j^+}(t)$ must take the following form:

$$F_{R_j^+}(t) = 1 - \sum_{k=j}^{K} \sum_{i=1}^{m_k} \beta_{i,k} A(k-1,\alpha_{i,k}) G_{k-j}(j,\alpha_{i,k}) \quad \text{for } t \geq 0 \tag{2-7-3}$$

and from (2-7-2) $G_k(j,s)$ satisfies the following recursive relation:

$$G_0(j,s) = e^{-st}$$

$$G_k(j,s) = A(j+k-2,s)G_{k-1}(j,s)$$

$$- \sum_{i=1}^{m_{j+k-1}} \beta_{1,j+k-1}A(j+k-2,\alpha_{i,j+k-1})G_{k-1}(j,s+\alpha_{i,j+k-1}) \tag{2-7-4}$$

Similarly, from equation (2-4-3) and (2-7-3) we have

$$F_{H_j}(t) = F_{S_j}(t) - \sum_{k=j}^{K} \sum_{i=1}^{m_k} \beta_{i,k}A(k-1,\alpha_{i,k})G'_{k-j-1}(j+1,\alpha_{i,k}) \tag{2-7-5}$$

where $G'_{k-j-1}(j+1,\alpha_{i,k}) = G_{k-j-1}(j+1,\alpha_{i,k})F_{S_j}(t)$.

Therefore $G'_k(j+1,s)$ satisfies the recursion given in (2-7-4) except that it starts with $G'_0(j+1,s) = G_0(j+1,s)F_{S_j}(t)$.

The mean interdemand time is obtained by substituting (2-7-3) for j=3 into (2-5-5)

$$E[H_1] = \mu + \sum_{k=3}^{K} \sum_{i=1}^{m_k} \beta_{i,k}A(k-1,\alpha_{i,k})E''_{k-3}(\alpha_{i,k}) \tag{2-7-6}$$

where μ and $E_k''(s)$ are defined as in (2-6-25) and (2-6-26), respectively. This quantity can be calculated only after $A(k-1,s)$ is found. To do this we need to define

$$B_{j,\ell} = \int_0^\infty F_{R_j^+}(x)\alpha_{\ell,j-1}e^{-\alpha_{\ell,j-1}x}\,dx \quad \text{for } j=2,\ldots,K$$

(2-7-7)

Substitute (2-7-3) into (2-7-7) to obtain

$$B_{j,\ell} = \begin{cases} 1 - \sum_{k=j}^{K}\sum_{i=1}^{m_k} \beta_{i,k}A(k-1,\alpha_{i,k})E_{k-j}(\ell,j,\alpha_{i,k}) & \text{for } j=2,3\ldots,K \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}7\text{-}8)$$

where $E_k(\ell,j,s) = \int_0^\infty G_k(j,s)\alpha_{\ell,j-1}e^{-\alpha_{\ell,j-1}x}\,dx$;

$E_k(\ell,j,s)$ satisfies the recursion given in (2-7-4) except that it starts with $E_0(\ell,j,s) = \alpha_{\ell,j-1}/(\alpha_{\ell,j-1}+s)$.

2.7.1. The Approximation Method

Substitution of (2-7-1) into (2-6-28) for $j = 2,3,\ldots,K$ gives

$$F_{I_j}(t) = 1 - \sum_{i=1}^{m_{j-1}} B_{j,i}\beta_{i,j-1}e^{-\alpha_{i,j-1}t} \quad \text{for } t \geq 0 \qquad (2\text{-}7\text{-}9)$$

Substitution of (2-7-9) into (2-6-6) gives, for $s \neq 0$,

$$A(j,s) = \begin{cases} 1 - \sum_{i=1}^{m_{j-1}} B_{j,i}\beta_{i,j-1}s/(s+\alpha_{i,j-1}) & \text{for } j=2,3,\ldots,K \\ 1 & \text{otherwise} \end{cases} \quad (2\text{-}7\text{-}10)$$

The system of equations defined by (2-7-8) and (2-7-10) must be solved to determine the value of $B_{j,i}$; $j=2,\ldots,K-1$ and $i=1,2,\ldots,m_j$. The system throughput rate is then obtained from equation (2-7-6) where $A(j,s)$ is defined as in (2-7-10). The number of equations that need to be solved here is $\sum_{i=2}^{K-1} m_i$, which means it grows linearly with the number of stages in the service time distributions used. As a comparison in the Markov model this number grows polynomially of degree K with the number of stages.

Using this method we calculated the normalized throughput rate of a balanced line for the case of m-stage general Erlang service times for up to K=8 and m=4. The results are shown in figure 2.2, where r_N is graphed as a function of η_S^2 (squared coefficient of variation of service time S) with K as a parameter. Since the throughput rate of a balanced production line is predominately determined by the first two moments of the service time distribution (Muth [1977]), this graph can be considered, for all practical purposes, to be a distribution-free property of a serial production line.

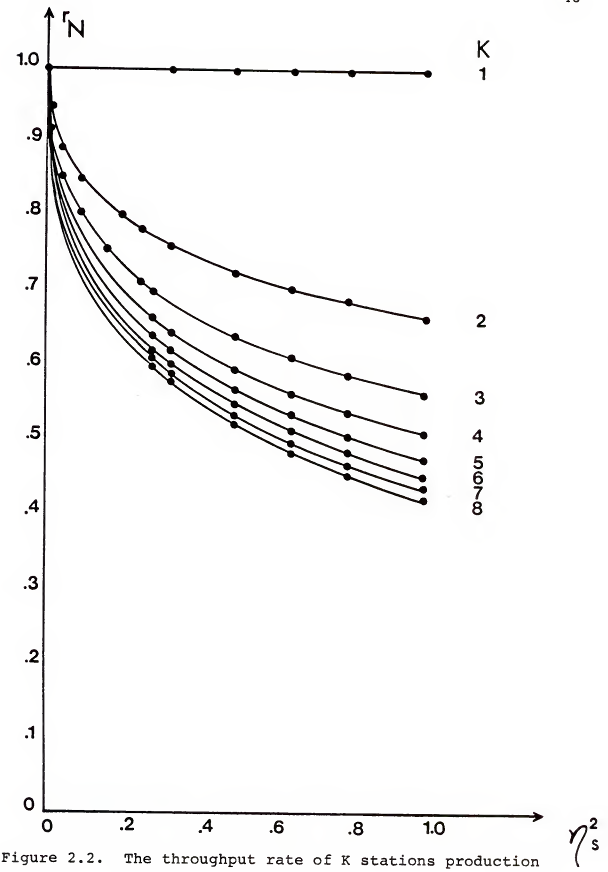


Figure 2.2. The throughput rate of K stations production line as a function of η_S^2: Erlang Case.

2.7.2. The Exact Method

Similar to what we did for the exponential case, if the line is balanced, i.e. if $\alpha_{ij} = \alpha_i$, $\beta_{ij} = \beta_i$ and $m_j = m$ for all $j \epsilon \kappa$, we have

$$A(2,s) = 1 - \sum_{i=1}^{m} B_{2,i}\beta_i s/(s+\alpha_i)$$

$$A(3,s) = 1 - \sum_{i=1}^{m} B_{3,i}\beta_i s/(s+\alpha_i) -$$

$$\left(\sum_{i=1}^{m} B_{2,i}\beta_i C_i s \ (s+\alpha_i)\right) \left(\sum_{j=1}^{m} \beta_j \alpha_j/(s+\alpha_j)\right).$$

where

$$C_i = \int_0^\infty e^{-\alpha_i x} dF_{H_3}(x).$$

The expression for A(i,s) when i > 3 grows very quickly in complexity. We are still able to derive the expression for A(4,s), beyond that things appear hopelessly complicated. Recall from equation (2-7-6) that A(i,s), i = 2, . . ., K-1 are required to compute the throughput rate of a K-station production line. For this reason the computer program was written to handle cases with $K \leq 4$. The numerical results of K = 4 shows that for the case $\eta_S = 1/\sqrt{2}$ the difference

between the throughput rate values calculated using the exact method and the approximation method is a half of the difference for the case $\eta_S = 1$. Since the approximation method is good enough for the case $\eta_S = 1$, we may say that for $\eta_S < 1$ it is sufficient to use the approximation method.

2.8. An Empirical Formula for the Throughput Rate of a Balanced Production Line

The throughput rate of a paced production line is given by

$$r_L = \frac{1}{E[\max(S_1,S_2,\ldots,S_K)]} \tag{2-8-1}$$

Muth [1973] shows that (2-8-1) is a lower bound for the throughput rate of the unpaced model of the same production line.

An explicit expression for the lower bound (2-8-1) in the case of identical and uniformly distributed service times with mean value equal to one is derived in Muth [1973]. The expression is

$$r_L = \frac{1}{1 + \sqrt{3}\ \eta_S \frac{K-1}{K+1}} \tag{2-8-2}$$

The uniform distribution was chosen because it covers a wide range of η_S; that is $0 \leq \eta_S \leq 1/\sqrt{3}$. However, formula (2-8-2) is intended to be used for all values of η_S.

The relative difference between the lower bound (2-8-2) and the actual throughput rate is a monotonically increasing function of η_S. This means that the difference is bigger for a bigger value of η_S. We compare the values of throughput rate calculated using (2-8-2) with the values of throughput rate calculated in previous sections using general Erlang service time. For K = 3 with $\eta_S = 1/\sqrt{3}$, the difference is about 2.2% and with $\eta_S = 1$, the difference is about 5%.

Considering the form of (2-8-2) we assume that an improved approximation for the actual normalized throughput rate should be of the form

$$r_N = \frac{1}{1 + f(\eta_S,K)} \qquad (2\text{-}8\text{-}3)$$

where $f(\eta_S,K)$ is a function with the following properties:

- $f(0,K) = 0$
- $f(\eta_S,1) = 0$
- nondecreasing with respect to η and K.

We assume further that $f(\eta_S,K)$ in (2-8-3) has the following form:

$$f(\eta_S,K) = a\eta_S \frac{K-1}{K+1+b\eta_S} \qquad (2\text{-}8\text{-}4)$$

The form of (2-8-3) together with (2-8-4) is very similar to the form (2-8-2) except that there is an additional term $b\eta$ in the denominator of $f(\eta_S,K)$. This additional term serves

as a correction factor that will compensate the difference between the lower bound and the actual value which is bigger for bigger η_S.

The task is now to determine the values of a and b in (2-8-4) from the numerical values of r_N obtained by the method of previous sections. From each value of r_N we computed $f(\eta_S, K)$ using equation (2-8-3) and fitted these values of $f(\eta_S, K)$ into equation (2-8-4) to get the values of a and b that minimize the error. We obtained

$$a = 1.67$$

$$b = 0.31.$$

Hence

$$r_N = \frac{1}{1 + 1.67\, \eta_S \dfrac{K-1}{K+1+.31\eta_S}}$$

In this curve fitting we used $K = 3, 4, \ldots, 10$ and $\eta_S = 1/\sqrt{3}, 1/\sqrt{2}, 1$. The maximum error obtained in these ranges of values of K and η_S is about 1%. For values of η_S that are less than $1/\sqrt{3}$, we expect to get a smaller error since this is the case for $K = 3$.

The absolute throughput rate r is obtained by dividing r_N with the mean service times; that is

$$r = r_N/\mu$$

where μ is the mean service time.

CHAPTER 3

PRODUCTION LINES WITH STATION BREAKDOWN

3.1. Introduction

In this chapter we treat the case where stations are subject to breakdown and repair. For this case, the following assumptions are made. The number of breakdowns of a station during a fixed time interval is a random variable. A station can break down only while it is actively servicing an item (i.e. in the busy state). When a station breaks down, the service is interrupted and a repair is begun. The item in progress remains in the station. Service is resumed on the item in progress after the repair has been completed. Breakdown of a station does not cause a shutdown of the entire line, but may affect adjacent stations through starving and blocking. Under the above assumptions, the period of time during which a station is broken down will always be nested in a service period for a single item.

The key idea of the following development is that the total time elapsed from the beginning to the completion of a service, including any repair periods that may intervene, is treated as a single random variable. This period (denoted by V) will be called the virtual service period. Any analysis method applicable to a model that does not consider

breakdown and uses as input the statistics of the service times may therefore be applied to the model including breakdown, by using the statistics of the virtual service period instead of the statistics of the service times. More specifically, if the throughput rate r_o of the production line model without breakdown is some function f of the distributions $F_{S_1}(t)$, $F_{S_2}(t)$, . . ., $F_{S_K}(t)$ of the station service times, namely

$$r_o = f \ [F_{S_1}(t), F_{S_2}(t), \ldots, F_{S_K}(t)]$$

then the throughput rate r_1 of the same model including breakdown is

$$r_1 = f \ [F_{V_1}(t), F_{V_2}(t), \ldots, F_{V_K}(t)]$$

where $F_{V_i}(t)$, i = 1, 2, . . ., K is the distribution function of the virtual service period of station i.

The chapter is organized as follows. In section 3.2 we derive a general formula for the transform of $F_V(t)$. In section 3.3 we specialize this formula to several models of the breakdown counting process. In section 3.4 we consider the Poisson breakdown process in more detail. We also state conditions under which the numerical procedures developed in Chapter 2 are applicable to the present model. Finally, in section 3.5 we suggest two approximation methods.

3.2. Derivation of the Transform of the Distribution of Virtual Service Period

Treated as a function of the parameter τ, the virtual service time $V(\tau)$ is a stochastic process that represents the total time required to achieve τ units of active service. We have that

$$V(\tau) = \tau + Q(\tau) \tag{3-2-1}$$

where $Q(\tau)$ is a process called excess time, introduced by Muth [1970, 1979a].

With S the service time of a station, the virtual service period of that station becomes

$$V(S) = S + Q(S) \tag{3-2-2}$$

where $Q(S)$ represents the accumulated time spent on repair during that service period (see Figure 3.1).

The excess time $Q(\tau)$ is a random sum of breakdown periods. We define $N(\tau)$ to be the number of breakdowns in the interval $(0,\tau)$. Its generating function is

$$G(\tau,z) \triangleq \sum_{n=0}^{\infty} z^n \, p(\tau,n) \tag{3-2-3}$$

where $\quad p(\tau,n) \triangleq P[N(\tau)=n].$

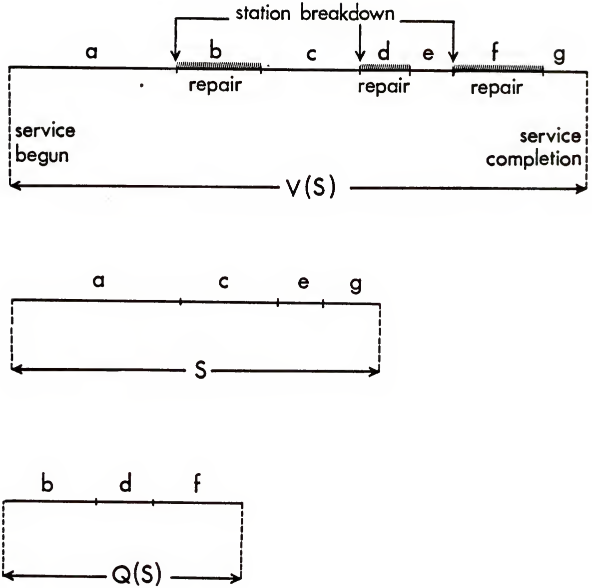


Figure 3.1. A particular realization of Q(S) and V(S).

We let T_j to be the repair time following the j-th breakdown. We assume that $T_1, T_2, \ldots, T_{N(\tau)}$ are independently and identically distributed random variables, each of them independent of $N(\tau)$. It then follows that

$$Q(\tau) = \sum_{n=0}^{N(\tau)} T_n \tag{3-2-4}$$

where $T_0 \triangleq 0$. Using conditioning on $N(\tau)$ the distribution function of $Q(\tau)$ is obtained from (3-2-4) as

$$F_{Q(\tau)}(x) = \sum_{n=0}^{\infty} F_T^{(n)}(x) \quad p(\tau,n) \tag{3-2-5}$$

where $F_T^{(n)}(x)$ is the n-fold convolution of $F_T(x)$, and $F_T{}^{(0)}(x) \triangleq 1$ for $x \geq 0$. Let $F^*(s)$ be the Laplace-Stieltjes transform of $F(\tau)$, then

$$F^*_{Q(\tau)}(s) = E[e^{-sQ(\tau)}] = \sum_{n=0}^{\infty} [F^*_T(s)]^n \; p(\tau,n) \tag{3-2-6}$$

or, using (3-2-3)

$$F^*_{Q(\tau)}(s) = G(\tau, F^*_T(s)) \tag{3-2-7}$$

To simplify notation we now let $V \triangleq V(S)$. The Laplace-Stieltjes transform $F_V^*(s)$ of the distribution function of V is obtained in two steps. The first step leads to the transform of the distribution of the process $V(\tau)$. From (3-2-1) and (3-2-6) we see that

$$F_{V(\tau)}^*(s) = E[e^{-s(\tau+Q(\tau))}] = e^{-s\tau}\ G(\tau, F_T^*(s)).$$

In the second step we condition on $S = \tau$ to obtain the distribution of V. Since

$$F_V(x) = \int_{\tau=0}^{\infty} F_{V(\tau)}(x)\ dF_S(\tau)$$

it follows that

$$F_V^*(s) = \int_0^{\infty} e^{-s\tau}\ G(\tau, F_T^*(s))\,dF_S(\tau) \tag{3-2-8}$$

This is a completely general formula relating the distribution of the virtual service period to the distribution of the service time, the distribution of the repair time and the distribution of the breakdown counting process. In order to reduce the integral in (3-2-8) to an algebraic expression, specific assumptions need to be made about the distributions of these quantities. Moreover, in order to compute the throughput rate using the numerical procedures

developed in Chapter 2, we must require that $F_V(t)$ be of a special phase type distribution. The question therefore arises as to what restrictions must be placed on the distributions of S, T and $N(\tau)$ in (3-2-8) in order to achieve this objective. In the process of specializing (3-2-8) we may begin by first selecting a specific distribution for S or by first selecting a specific model for the breakdown process $N(\tau)$. The latter case will be treated in section 3.3.

3.2.1. The Case Where Service Time is of Special Phase Type

In the less complex model of Chapter 2 the service distribution was restricted to that of special phase type, for the reason just mentioned. It is therefore appropriate to apply the same restriction to this model. We then have

$$F_S(\tau) = 1 - \sum_{i=1}^{m} \beta_i \, e^{-\alpha_i \tau}$$

and equation (3-2-8) becomes

$$F_V^*(s) = \sum_{i=1}^{m} \alpha_i \beta_i \int_0^{\infty} e^{-s\tau} \, G(\tau, F_T^*(s)) e^{-\alpha_i \tau} d\tau$$

$$= \sum_{i=1}^{m} \alpha_i \beta_i \, \bar{G} \, (s + \alpha_i, F_T^*(s)). \qquad (3\text{-}2\text{-}9)$$

where $\bar{G}(s,z)$ is the Laplace transform of $G(\tau,z)$. This equation will be further specialized in section 3.3 to a model of $N(\tau)$ that has a closed form expression for $\bar{G}(s,z)$.

3.2.2. The Simplest Model

As an example of how (3-2-8) may be reduced to an algebraic expression we treat the simplest model which arises when S and T are exponentially distributed with rates α and β and when $N(\tau)$ is a Poisson process with rate λ. For this case, we have

$$F_S(\tau) = 1-e^{-\alpha\tau}$$

$$F_T^*(s) = \frac{\beta}{s+\beta}$$

$$G(\tau,z) = e^{-\lambda\tau(1-z)}$$

Substitution into (3-2-8) yields

$$F_V^*(s) = \alpha \int_0^\infty e^{-s\tau}\, e^{-\lambda(1-F_T^*(s))\tau}\, e^{-\alpha\tau}d\tau$$

$$= \frac{\alpha}{s+\alpha+\lambda(1-\frac{\beta}{s+\beta})}$$

$$= \frac{\alpha(s+\beta)}{s^2+(\alpha+\beta+\lambda)s+\alpha\beta} \qquad (3\text{-}2\text{-}10)$$

The above expression was previously obtained by Altiok and Stidham [1983] in a direct manner by using the no-memory property of exponential random variables.

The poles of $F_V^*(s)$ can be obtained from equation (3-2-10) as

$$s_{1,2} = \frac{-(\alpha+\beta+\lambda) \pm \sqrt{(\alpha+\beta+\lambda)^2 - 4\alpha\beta}}{2} \qquad (3\text{-}2\text{-}11)$$

After some manipulation we have

$$s_{1,2} = \frac{-(\alpha+\beta+\lambda) \pm \sqrt{(\alpha-\beta-\lambda)^2 + 4\alpha\lambda}}{2} \qquad (3\text{-}2\text{-}12)$$

From equations (3-2-11) and (3-2-12) we can see that s_1 and s_2 are always real, distinct and negative. Therefore $F_V^*(s)$ always has real, distinct and negative poles for all combination values of α, β and λ. This means, for the simplest model, $F_V(t)$ is always a special phase type distribution. In section 3.4 we discuss more general models for which $F_V(t)$ is still a special phase type distribution.

3.3. Model for $N(\tau)$

In this section we select specific models for the breakdown counting process and substitute them into the formulation (3-2-8) to obtain algebraic expressions of

$F_V^*(s)$. We also state the condition under which $F_V^*(s)$ is of phase type.

3.3.1. Poisson Process

The simplest model is obtained when $N(\tau)$ is a Poisson process. For this model we have

$$F_U(\tau) = 1 - e^{-\lambda\tau}$$

where U is the time between breakdowns and λ is the station breakdown rate. Also

$$p(\tau,n) = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$$

and

$$G(\tau,z) = e^{-\lambda\tau(1-z)} \qquad (3\text{-}3\text{-}1)$$

Substitution of (3-3-1) into (3-2-8) yields

$$F_V(s) = \int_0^\infty e^{-s\tau} e^{-\lambda\tau(1-F_T^*(s))} d\, F_S(\tau)$$

$$= F_S^*(s + \lambda - \lambda F_T^*(s)) \qquad (3\text{-}3\text{-}2)$$

The same expression was obtained by Gaver [1962] as the total service for a customer with low priority in an M/G/1 queueing system with two priorities. The repair time T in (3-3-2) corresponds to the service time of a customer with high priority.

3.3.2. Renewal Counting Process

If the sequence of time between breakdowns U_1, U_2, . . . is independently and identically distributed, then this sequence forms an ordinary renewal process and $N(\tau)$ is the corresponding renewal counting process. An ordinary renewal process in the context of our model implies that the beginning of a virtual service period coincides with a renewal point, which means that a repair has just been completed. In reality the beginning of a virtual service period will fall randomly between renewal points. Hence an equilibrium renewal process is a more appropriate model.

In an equilibrium renewal process U_2, U_3, . . . are independently and identically distributed as U, and U_1 is the forward recurrence time of the process with distribution function

$$F_{U_1}(\tau) = \frac{1}{E[U]} \int_0^{\tau} (1 - F_U(x))dx \qquad (3\text{-}3\text{-}3)$$

The probability law of $N(\tau)$ is linked to the probability law of U and U_1 by the well known relation

$$p(\tau,n) = \begin{cases} 1 - F_{U_1}(\tau) & \text{for } n=0 \\ F_{U_1}(\tau) * F_U^{(n-1)}(\tau) * (1 - F_U(\tau)) & \text{for } n=1,2,\ldots \end{cases} \tag{3-3-4}$$

where * denotes convolution.

Taking the Laplace transform of (3-3-4) gives

$$\bar{p}(s,n) = \begin{cases} \frac{1}{s}(1 - F_{U_1}^*(s)) & \text{for } n=0 \\ \frac{1}{s} F_{U_1}^*(s)\ (F_U^*(s))^{n-1}\ (1-F_U^*(s)) & \text{for } n=1,2,\ldots \end{cases} \tag{3-3-5}$$

Applying the definition (3-2-3) to (3-3-5) we obtain the transform of the generating function of $N(\tau)$ as

$$\bar{G}(s,z) = \frac{1}{s}\left(1-F_{U_1}^*(s) + \frac{zF_{U_1}^*(s)\ (1-F_U^*(s))}{1-z\ F_U^*(s)}\right)$$

or

$$\bar{G}(s,z) = \frac{1-F_{U_1}^*(s) + z\ (F_{U_1}^*(s) - F_U^*(s))}{s\ (1-z\ F_U^*(s))} \tag{3-3-6}$$

where from (3-3-3)

$$F^*_{U_1}(s) = \frac{1-F^*_U(s)}{E[U]}$$

Thus, after some manipulation we have

$$\bar{G}(s,z) = \frac{1}{s} - \frac{(1-z)(1-F^*_U(s))}{sE[U](1-z\,F^*_U(s))} \qquad (3\text{-}3\text{-}7)$$

If U_1 is identical to U then we have an ordinary renewal process and (3-3-6) reduces to

$$\bar{G}(s,z) = \frac{1-F^*_U(s)}{s(1-zF^*_U(s))}$$

Substitution of (3-3-7) into (3-2-9) yields

$$F^*_V(s) = \sum_{i=1}^{m} \alpha_i\beta_i \left(\frac{1}{s} - \frac{(1-F^*_T(s))\;(1-F^*_U(s+\alpha_i))}{sE[U](1-F^*_T(s)\;F^*_U(s+\alpha_i))}\right) \qquad (3\text{-}3\text{-}8)$$

From this equation we can see that if $F^*_U(s)$ and $F^*_T(s)$ are rational functions, then $F^*_V(s)$ is also a rational function.

3.3.3. Phase-Renewal Counting Process

This special case of a renewal process is obtained by letting the time between breakdowns U be a phase type random variable. We again consider the case of an equilibrium process. For simplicity, assume that U is a continuous random variable. In order to make subsequent results notationally compact, the distribution function of U is expressed as

$$F_U(\tau) = 1 - q\, e^{C\tau} u \qquad (3\text{-}3\text{-}9)$$

where q is a row probability vector, u is a column vector whose elements are 1, and C is a matrix with special properties as discussed in appendix A. The Laplace-Stieltjes transform of $F_U(\tau)$ is

$$F_U^*(s) = q\,(-C)\,(sI-C)^{-1}u \qquad (3\text{-}3\text{-}10)$$

From (3-3-3) and (3-3-9) we obtain

$$F_{U_1}^*(s) = \frac{q\,(sI-C)^{-1}u}{E[U]} \qquad (3\text{-}3\text{-}11)$$

By comparing (3-3-10) with (3-3-11) it is apparent that U_1 is also of phase type and that $F_{U_1}(\tau)$ consists of a combination of exactly the same exponential functions that are present in $F_U(\tau)$. Therefore we must be able to write

$$F_{U_1}^{*}(s) = \pi\ (-C)\ (sI-C)^{-1}u \tag{3-3-12}$$

and

$$F_{U_1}(\tau) = 1-\pi\ e^{C\tau}u \tag{3-3-13}$$

where π is a row probability vector. Comparing (3-3-11) and (3-3-12) we see that

$$\pi\ (-C) = \frac{q}{E[U]}$$

Multiplying the above equation by the matrix uq yields

$$\pi\ (-C)\ uq = \frac{quq}{E[U]} = \frac{q}{E[U]}$$

and therefore

$$\pi\ (-C) = \pi\ (-C)\ uq.$$

The result is the defining equations for the vector π, namely

$$\begin{cases} \pi\ (C-Cuq) = 0 \\ \pi\ u \qquad\quad = 1. \end{cases}$$

This equation for $F_{U_1}(\tau)$ was derived by Neuts [1981] using a different method.

Substitution of $F_{U_1}(\tau)$ and $F_U(\tau)$ into (3-3-4) and taking the Laplace transform of the resulting expression yields

$$\bar{p}(s,n) = \begin{cases} \pi\ (sI-C)^{-1}u & \text{for } n=0 \\ \pi\ (sI-C)^{-1}(-C)u[q(sI-C)^{-1}(-C)u]^{n-1}q(sI-C)^{-1}u & \text{for } n=1,2... \end{cases}$$

Writing B = -Cuq, this equation becomes

$$\bar{p}(s,n) = \pi\ ((sI-C)^{-1}B)^n(sI-C)^{-1}u \qquad \text{for } n = 0,1,2... \tag{3-3-14}$$

Substitution in (3-2-3) yields

$$\bar{G}\ (s,z) = \pi\ (I-z(sI-C)^{-1}B)^{-1}(sI-C)^{-1}u$$

Since B and (sI-C) commute, we have

$$\bar{G}\ (s,z) = \pi\ (sI-C-zB)^{-1}u$$

$$= \pi\ (sI-(C+zB))^{-1}u \tag{3-3-15}$$

The generating function of $N(\tau)$ is now obtained by taking the inverse Laplace transform of (3-3-15)

$$G(\tau,z) = \pi\, e^{(C+zB)\tau} u \tag{3-3-16}$$

or

$$G(\tau,z) = \pi\, e^{C(I-zuq)\tau} u$$

If we wanted to use an ordinary renewal process we would simply replace π in the above equation with q. Equation (3-3-16) was also derived by Neuts [1981] using a different method.

As a final step we determine the transform of the distribution of virtual service period. Substitution of (3-3-16) into (3-2-8) yields

$$F_V^*(s) = \int_0^\infty e^{-s\tau}\, \pi\, e^{(C+F_T^*(s)B)\tau} u \; dF_S(\tau)$$

$$= \pi \int_0^\infty e^{-(sI-C-F_T^*(s)B)\tau}\, dF_S(\tau)\; u$$

$$= \pi\, F_S^* \,(sI-C-F_T^*(s)B)\; u \tag{3-3-17}$$

It can be seen from (3-3-17) that $F_V^*(s)$ will be a rational function if $F_T^*(s)$ and $F_S^*(s)$ are rational functions.

3.4. The Case of Poisson Breakdown Process

It is reasonable to assume that the station time between breakdowns has no-memory, which means that the breakdown counting process is a Poisson process. For this case, we showed that

$$F_V^*(s) = F_S^*(s + \lambda - \lambda F_T^*(s)) \tag{3-3-2}$$

As discussed in section 3.2 we wish to have $F_V^*(s)$ of special phase type so that the procedures for calculating the throughput rate developed in the previous chapter can be applied. In this section we determine those special models of $F_S^*(s)$ and $F_T^*(s)$ that lead to $F_V^*(s)$ of special phase type.

From the structure of (3-3-2) we conclude first that $F_S^*(s)$ needs to be a rational function with distinct poles only. Thus, writing

$$F_S^*(s) = \sum_{i=1}^{m} \frac{\alpha_i \beta_i}{(s+\alpha i)}$$

we obtain

$$F_V^*(s) = \sum_{i=1}^{m} \frac{\alpha_i \beta_i}{s+\alpha_i+\lambda-\lambda F_T^*(s)} \tag{3-4-1}$$

Next we conclude that if $F_T^*(s)$ is a rational function then $F_V^*(s)$ becomes a rational function. However the poles of $F_V^*(s)$ are not necessarily all real even though all α_i are real and $F_T^*(s)$ has only real poles. It is too difficult to investigate the behavior of the poles of $F_V^*(s)$ for a general model of $F_T(t)$. Instead we will investigate the poles of $F_V^*(s)$ for two specific models of $F_T(t)$, one is a hyperexponential distribution, the other is a two-stage general Erlang distribution, that is, a hypoexponential distribution. These two distributions are the simplest form of special phase type distributions that permit the value of the squared coefficient of variation to be different from 1. The coefficient of variation of a hyperexponential distribution is greater than 1 and that of a two-stage general Erlang distribution is between $\sqrt{0.5}$ and 1.

Case 1: $F_T(t)$ is a hyperexponential distribution

For this case we have the following result:

Result 3.1: If T is a hyperexponential random variable, then all poles of $F_V^*(s)$ are real, distinct and negative.

Proof: If T is a hyperexponential random variable, then $F_T^*(s)$ has the form

$$F_T^*(s) = \sum_{i=1}^{n} \frac{\gamma_i \delta_i}{(s+\delta_i)} \tag{3-4-2}$$

where n is finite; γ_i, $\delta_i > 0$ and $\sum_{i=1}^{n} \gamma_i = 1$.

For convenience, let the δ_i be assigned such that $\delta_1 < \delta_2 < \cdot \cdot \cdot < \delta_n$. Consider first the j-th term of the summation in equation (3-4-1), which is

$$h_j(s) = \frac{\alpha_j \beta_j}{(s+\alpha_j+\lambda-\lambda \sum_{i=1}^{n} \frac{\gamma_i \delta_i}{(s+\delta_i)})} = \frac{\alpha_j \beta_j}{g_j(s)}$$

The denominator of this expression is an increasing function of real s for $-\delta_{i+1} < s < -\delta_i$ (see figure 3.2) and therefore has exactly one real zero in each such interval. It follows from a similar argument that there is exactly one real zero in the interval $(-\infty, -\delta_n)$ and exactly one real zero in the interval $(-\delta_1, 0)$. Since $g_j(s)$ is of degree (n+1) there cannot be any other zeros. Consider next the zeros of $g_k(s)$, where $k \neq j$. The function $g_k(s)$ differs from $g_j(s)$ because of the parameter ${}^2{}_k$. Thus if ${}^2{}_k \neq {}^2{}_j$ then for each of the n+1 nonoverlapping intervals on the s-axis, the zero of $g_k(s)$ is different from the zero of $g_j(s)$. This case is shown in figure 3.2. Hence if the ${}^2{}_j$, j = 1, 2, ..., m are distinct, then the poles of $F_V^*(s)$ are distinct.

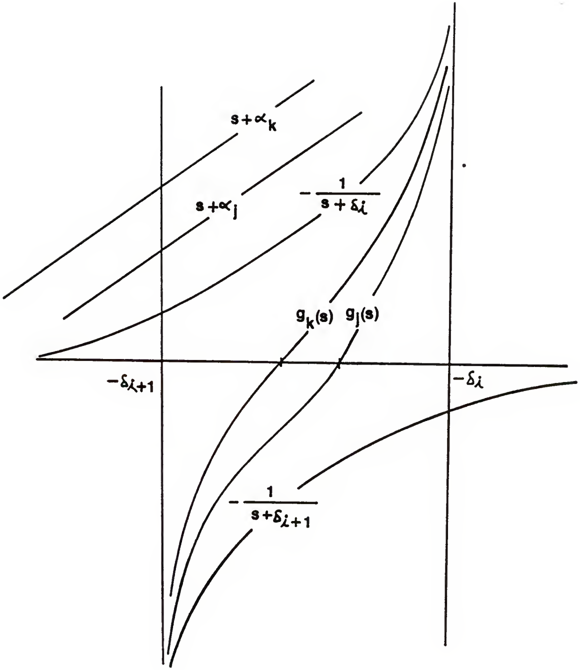


Figure 3.2. The graphs of $g_j(s)$ and $g_k(s)$ for $-\delta_{i+1} < s < -\delta_i$.

This result guarantees that the numerical procedures for calculating the throughput rate developed in the previous chapter can be used as long as $F_S(t)$ is of special phase type and $F_T(t)$ is hyperexponential. Note that if we wish to prescribe only the first two moments of the repair time, then we need only two terms in the sum (3-4-2).

<u>Case 2: $F_T(t)$ is a two-stage general Erlang distribution</u>

If T is two-stage general Erlang random variable, then $F_T^*(s)$ has the form

$$F_T^*(s) = \frac{\delta_1 \delta_2}{(s+\delta_1)(s+\delta_2)}$$

Consider first the j-th term of the summation in equation (3-4-1), which is

$$h_j(s) = \frac{\alpha_j \beta_j}{s+\alpha_j+\lambda-\lambda \dfrac{\delta_1 \delta_2}{(s+\delta_1)(s+\delta_2)}}$$

The poles of $h_j(s)$ are the roots of the characteristic equation

$$1 + \lambda \frac{s(s+\delta_1+\delta_2)}{(s+\alpha_j)(s+\delta_1)(s+\delta_2)} = 0 \qquad (3\text{-}4\text{-}3)$$

We wish to determine the range values of λ, α_j, δ_1 and δ_2 for which these poles are real. A convenient tool for investigating the roots of an equation is the root locus plot. A brief explanation of the root locus method follows. Let equation (3-4-3) be written as

$$1 + \lambda\, G(s) = 0$$

where λ may take values in $(0, \infty)$. The roots of this equation are the values of s that satisfy

$$s = G^{-1}(-1/\lambda)$$

The root locus is a one dimensional curve that maps the negative real axis $-1/\lambda$ into the s-plane via the mapping function $G^{-1}(\cdot)$ and parameterized by the values of λ. Of interest here are those sections of the root locus that lie on the real s-axis. A reference on the methodology of root locus plotting is Dorf [1967].

In applying the root locus method to equation (3-4-3) it turns out that two separate cases need to be considered here. In both cases, the root locus has three distinct branches corresponding to the three solutions of (3-4-3).

Case 1: $\alpha_j > \delta_1 + \delta_2$

The root locus of this case is shown in figure 3.3.a. It is seen that the poles of $F_V^*(s)$ are always real, distinct and negative.

Case 2: $\alpha_j < \delta_1 + \delta_2$

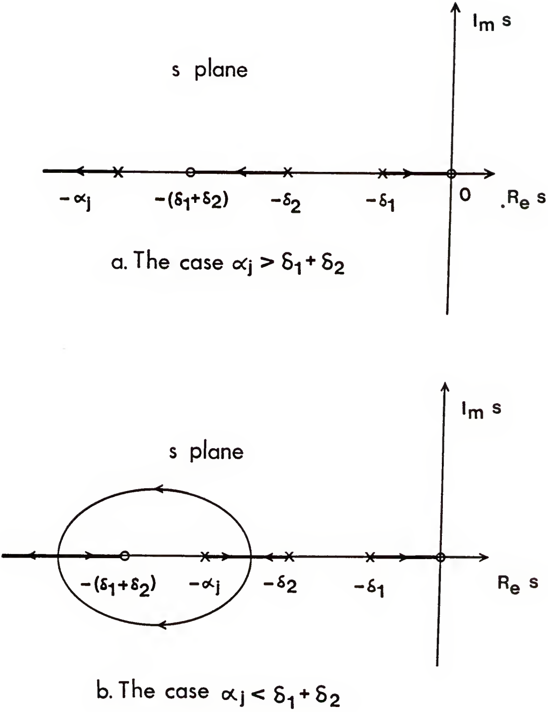
The root locus of this case is shown in figure 3.3.b. The figure shows the case $\alpha_j > \delta_2$. If $\delta_1 < \alpha_j < \delta_2$ or $\alpha_j < \delta_1$ the structure of the root locus plot remains unchanged; and only the order of α_j, δ_2 and δ_1 on the real axis is changed. In this case there is a range of values of λ in which two of the poles of $F_V^*(s)$ are complex conjugate. The range of values of λ that still give real poles is obtained as follows. From equation (3-4-3) represent λ as a function of real s

$$\lambda = \lambda(s)$$

and then find the real solutions of the equation

$$\frac{d\lambda(s)}{ds} = 0 \qquad (3\text{-}4\text{-}4)$$

In our case the above equation has four roots, two of which are real and the other two are complex conjugate. While it is easy to find these two real roots numerically, their closed form expressions are not so simple. Let the real



X : $\lambda = 0$ O : $\lambda = \infty$

Figure 3.3. The root locus of equation (3-4-3) with λ as the parameter.

solution of (3-4-4) be s_1 and s_2 and $\lambda(s_1) < \lambda(s_2)$, then the range values of λ that gives real roots are

$$0 \leq \lambda \leq \lambda(s_1) \quad \text{and} \quad \lambda \geq \lambda(s_2)$$

The following result is a consequence of the first case.

Result 3.2: If T is a two-stage general Erlang random variable and $\min_j \{\alpha_j\} \geq \delta_1 + \delta_2$, then the poles of $F_V^*(s)$ are all real, distinct and negative.

From the above discussion we see that the poles of $F_T^*(s)$ are not always real when $F_T(t)$ is a two-stage Erlang distribution. Obviously the possiblity of obtaining complex poles will be greater for more complex models of $F_T(t)$, such as three- or four-stage Erlang distributions.

3.5. Approximation Methods

From equation (3-3-17) we can see that the degree of complexity of the virtual service period distribution is the compounded effect of the complexities of the distribution of the service time, the repair time and the time between breakdowns. In practice these random variables do not have a simple distribution function, such as exponential, since their coefficients of variations are usually very different from 1. The transform of these distributions therefore may have a large number of poles. If this is the case then the virtual service period distribution becomes very complex, meaning that the poles of its transform are very likely to

be nonreal or nondistinct. Even when it is reasonable to assume that the time between breakdowns is exponential, the service time and the repair time may still have complex distribution functions, and it was shown that for this simple model of $N(\tau)$ the poles of the transform of the virtual service period distribution are not always real and distinct. Even if all poles are real and distinct there may be so many of them as to make the numerical procedure inefficient.

Another difficulty is that in complex cases the primary information we have about the random variables service time, repair time and time between breakdowns usually is not on their distribution functions, but only limited on their first two moments.

To approach such complex cases, it is probably better to use efficient, but fairly accurate, approximation methods, rather than to insist on exact but cumbersome computations. Two such approximation methods are presented next. One is the "single breakdown approximation" and the other is the "two moment approximation."

3.5.1. Single Breakdown Approximation

In order for a system to be usable in practice, the probability of one or more breakdowns during any single service period should be small. As a consequence, the probability of more than one breakdown during any service period will be negligibly small. Therefore, it is reasonable to

assume that during a single service period, a station will break down at most once. Such an approximation was suggested by Muth [1979a].

Let U_1 be the time until the first breakdown with distribution function $F_{U_1}(t)$. It follows that the breakdown counting process under the single breakdown approximation has the form

$$P[N(\tau) = n] = \begin{cases} 1 - F_{U_1}(\tau) & \text{for } n=0 \\ F_{U_1}(\tau) & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}5\text{-}1)$$

This approximation is good only for the values of τ such that $P[N(\tau) = 0]$ is close to one. As a rule of thumb we can pick a value τ^* for which $P[N(\tau^*) = 0] = 0.95$ and say that the approximation is justified when $E[S] + 2\sqrt{Var(s)} < \tau^*$.

The generating function of $N(\tau)$ is obtained from (3-5-1) as

$$G(\tau,z) = 1 - F_{U_1}(\tau) + F_{U_1}(\tau)z \qquad (3\text{-}5\text{-}2)$$

If U_1 has the phase type distribution given under (3-3-13), then

$$G(\tau,z) = \pi\, e^{C\tau}u + z\,(1-\pi\, e^{C\tau}u). \qquad (3\text{-}5\text{-}3)$$

Substitution of the above expression into (3-2-8) yields

$$F_V^*(s) = \int_0^\infty e^{-s\tau}\ \pi\ e^{C\tau}u\ dF_S(\tau)$$

$$+\ F_T^*(s) \int_0^\infty e^{-s\tau}\ (1-\pi\ e^{C\tau}u)\ dF_S(\tau)$$

After some simplification we obtain

$$F_V^*(s) = F_T^*(s)\ F_S^*(s) + (1-F_T^*(s))\ \pi\ F_S^*(sI-C)u \qquad (3\text{-}5\text{-}4)$$

Close inspection of the above equation shows that the number of poles is not fixed but is less than or equal to $m(n + 1) + k$ where m and k are the number of poles of $F_S^*(s)$ and $F_T^*(s)$ and n is the size of matrix C. By comparison, the number of poles resulting from the exact formula (3-3-17) is $mn(k + 1)$. In most cases $mn(k + 1)$ is greater than $m(n + 1) + k$. Let $F_S^*(s)$ be a special phase type distribution with poles $-\alpha_i$, $i=1, \ldots, m$ and $F_T^*(s)$ be a special phase type distribution with poles $-\gamma_j$, $j=1, \ldots, k$. We then have that the poles of $F_V^*(s)$ are real and distinct if and only if α_i and γ_j are distinct for all i and j, and the n roots of the equation $|(s+\alpha_i)I - C| = 0$ are all real and distinct for all i and not equal to α_i or γ_j. We have the following result.

Result 3.3: If T and S are special phase type random variables with poles $-\alpha_i$ and $-\gamma_j$ where $\alpha_i \neq \gamma_j$ for all i and j, and the breakdown process is a Poisson process with rate λ such that $\alpha_i + \lambda \neq \gamma_j$ for all i and j, then the poles of $F_V^*(s)$ are all real and distinct.

Proof: If the breakdown counting process is a Poisson process with rate λ, equation (3-5-4) becomes

$$F_V^*(s) = F_T^*(s)\ F_S^*(s) + (1-F_T^*(s))\ F_S^*(s+\lambda)$$

and the proof is at hand.

Thus to attain distributions with real and distinct poles is easier with the approximation (3-5-4) than with the exact formula (3-3-17).

In some cases, however, the exact formula (3-3-17) is to be preferred over the approximation (3-5-4). As an example, suppose that the breakdown counting process is a Poisson process, the time to repair is exponential and the service time is of special phase type. If the exact formula is used then the number of poles will be 2m and according to result 3.1 all of them are real, distinct and negative. If the approximation formula is used then the number of poles will be 2m + 1 and all of them are real but not necessarily distinct. For the poles to be distinct, using result 3.3, the repair rate minus the breakdown rate must be not equal

to any poles of $F_S^*(s)$. This approximation method is useful only if the exact method fails, because it does not yield real poles, or if $mn(k+1) > m(n+1) + k$ and the approximation gives real and distinct poles. If the exact and the single breakdown approximation methods fail, then the two moment approximation, which will be given in the next section, should be used.

3.4.2. Two Moment Approximation

Muth [1977] computes and compares the throughput rate of three-station balanced production lines when all stations have Erlang service times and when all stations have uniform service times. The comparison is made with distributions whose first two moments are matched. The results show that the throughput rates over a range of values of the coefficient of variations are, for all practical purposes, identical. Since these two distributions differ considerably in their higher moments, the results indicate that the third and higher moments of the service time have negligible effect on the throughput rate. Applying this result to our case we conclude that the throughput rate of a production line subject to breakdown may be approximated by a method that requires as input only the first two moments of the virtual service periods.

Such an approximation is useful if the virtual service period V has a distribution function that is either very complex, meaning the transform has a large number of poles,

or if the transform has complex or nondistinct poles. The key idea of the approximation is to compute the first two moments of V directly from the parameters of S, T and $N(\tau)$, and then to compute the throughput rate, using the methodology of chapter 2, with the simplest and the most convenient distribution of service times that matches the first two moments of V.

First we need to establish general expressions that relate the first two moments of V to the parameters of S, T and $N(\tau)$. One way to obtain these expressions is by using the formula

$$E[V^i] = (-1)^i \frac{d^i F_V^*(0)}{ds^i}$$

where $F_V^*(s)$ is given by equation (3-2-8). Here we use a more direct approach as in Muth [1979a]. We have

$$V = S + Q(S) \tag{3-2-2}$$

or

$$V = S + \sum_{i=1}^{N(S)} T_i \tag{3-5-5}$$

Since the T_i are independently and identically distributed as T, taking the expected value of the two sides of equation (3-5-5) yields

$$E[V] = E[S] + \sum_{i=1}^{\infty} i\, E[T]\, P[N(S) = i]$$

or

$$E[V] = E[S] + E[T]\, E[N(S)]$$

By conditioning E[N(S)] on S we obtain

$$E[V] = E[S] + E[T] \int_0^{\infty} E[N(\tau)]\, dF_S(\tau) \qquad (3\text{-}5\text{-}6)$$

The second moment $E[V^2]$ is obtained by squaring equation (3-2-2), completing the square on the right hand side, and taking the expected value of all terms. The result is

$$E[V^2] = E[S^2] + 2E[SQ(S)] + E[Q(S)^2]$$

and, conditioning on S as before, we obtain

$$E[V^2] = E[S^2] + 2E[T] \int_0^\infty \tau \quad E[N(\tau)] \; dF_S(\tau) +$$

$$Var[T] \int_0^\infty E[N(\tau)] \; dF_S(\tau) + E[T]^2 \int_0^\infty E[N(\tau)^2] \; dF_S(\tau) \tag{3-5-7}$$

If $N(\tau)$ is a Poisson process, then we have the following formulas:

$$E[N(\tau)] = \frac{\tau}{E[U]}$$

$$E[N(\tau)^2] = \frac{\tau}{E[U]} + \frac{\tau^2}{E[U]^2}$$

Substitution of the above equations into (3-5-6) and (3-5-7) yields, after some simplifications,

$$E[V] = E[S] \; (1 + E[T]/E[U]) \tag{3-5-8}$$

$$E[V^2] = E[S^2] \; (1 + E[T]/E[U])^2$$

$$+ (1 + \eta_T^2) \; \frac{E[T] \; E[S]^2}{E[U]} \tag{3-5-9}$$

where η denotes coefficient of variation.

From these two equations we can see that we need to have information only about the first two moments of S, T,

and U in order to compute the first two moments of V. Here we are interested in the dependence of η_V^2 on η_S^2. This relation will let us select an appropriate distribution that matches the first two moments of V. After some simplification we obtain from (3-5-8) and (3-5-9) the following relation:

$$\eta_V^2 = \eta_S^2 + (1 + \eta_T^2)\ g(x,y) \tag{3-5-10}$$

where
$$g(x,y) = \frac{x^2 y}{(x+y)^2}$$

$$x = E[T]/E[S]$$

$$y = E[U]/E[S]$$

It is seen that η_V^2 is always greater than or equal to η_S^2. Figure 3.4 shows the contours of g(x,y) as x and y are varied from 0 to 5.0. Using these contours η_V^2 can be found easily.

Now we select the simplest model for the moment-matching distribution to be used in the calculation of the throughput rate. Depending on the value of the squared coefficient of variation of the virtual service period, two models of special phase type distribution functions can be selected. One is the hyperexponential distribution, for the

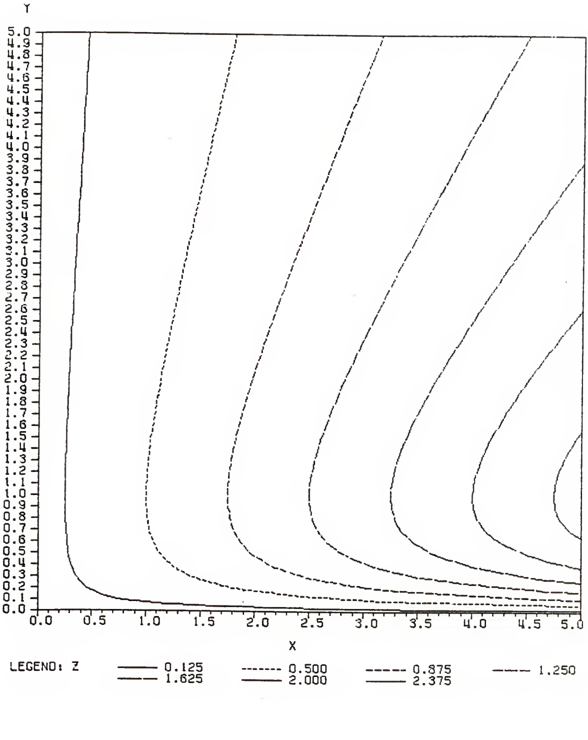


Figure 3.4. Contours of constant g(x,y).

case $\eta_V^2 > 1$, and the other is the hypoexponential, for the case $\eta_V^2 \le 1$.

Case 1: The virtual service period is hyperexponential

The simplest model is a mix of only two exponential functions, namely

$$F_V(t) = 1 - \beta e^{-\alpha_1 t} - (1-\beta)\, e^{-\alpha_2 t}$$

This model is sufficient to cover the range of all values of η_V^2 greater than 1. We must determine the parameters α_1, α_2 and β that satisfy the equations

$$\frac{\beta}{\alpha_1} + \frac{1-\beta}{\alpha_2} = E[V] \qquad (3\text{-}5\text{-}11)$$

$$\frac{\beta}{\alpha_1^2} + \frac{1-\beta}{\alpha_2^2} = E[V^2]/2 \qquad (3\text{-}5\text{-}12)$$

$$\beta_1,\ \alpha_1,\ \alpha_2 > 0$$

These two equations in three unknowns do not have a unique solution. In practice one may choose one parameter and then determine the other two from the above two equations. One way to do this is by imposing the following restriction:

$$\frac{\beta}{\alpha_1} = \frac{1-\beta}{\alpha_2} \tag{3-5-13}$$

This restriction means that each exponential term contributes exactly one-half of the mean value. From equations (3-5-11) through (3-5-13), we obtain

$$\beta = \frac{1}{2}\left(1 + \sqrt{\frac{\eta_V^2 - 1}{\eta_V^2 + 1}}\right)$$

$$\alpha_1 = \frac{2\beta}{E[V]}$$

and

$$\alpha_2 = \frac{1-\beta}{\beta}\ \alpha_1$$

<u>Case 2: The virtual service period is a hypoexponential</u>

The simplest model for this case is an m-stage general Erlang distributions with the smallest number of stages required to attain the desired coefficient of variation η_V^2. We thus have

$$F_V(t) = 1 - \sum_{i=1}^{m} \beta_i\ e^{-\alpha_i t}$$

where $$\beta_i = \frac{\prod_{j=1}^{m} \alpha_i}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (\alpha_j - \alpha_i)}$$

and m is the smallest integer greater than $\frac{1}{\eta_V^2}$.

The parameters α_i must satisfy the following equations:

$$\sum_{i=1}^{m} \frac{1}{\alpha_i} = E[V] \tag{3-5-14}$$

$$\sum_{i=1}^{m} \frac{1}{\alpha_i^2} = Var\ [V] \tag{3-5-15}$$

$$\alpha_i > 0 \qquad \text{for } i = 1, 2, \ldots, m$$

We have two equations with m unknowns. These equations do not have a unique solution. One way to cooose a suitable set of α_i is to impose the following restrictions:

$$\frac{1}{\alpha_{i+1}} - \frac{1}{\alpha_i} = \frac{aE[V]}{m} \qquad i = 1, 2, \ldots, m-1 \tag{3-5-16}$$

where a is some postive constant that needs to be determined so that equations (3-5-14) and equation (3-5-15) are

satisfied. From equations (3-5-14) through (3-5-16) we obtain

$$a = 2\sqrt{\frac{3(m\eta_V^2 - 1)}{m^2 - 1}}$$

and

$$\frac{1}{\alpha_i} = \frac{E[V]}{m}\left(1 + a\left(i - \frac{m+1}{2}\right)\right)$$

CHAPTER 4

THE BOWL PHENOMENON

4.1. Introduction

The problem of unbalancing a production line subject to certain constraints, which we refer to as constrained unbalancing, has attracted attention in almost 25 papers since it was introduced by Hillier and Boling [1966].

Hillier and Boling analyzed a three-station production line with exponential service times, in which two stations are identical, subject to the constraint that the sum of the mean service times of the three stations is constant. They find that the maximum throughput rate is obtained by decreasing the mean service time of the middle station and by increasing the mean service time of the two identical outside stations. Hillier and Boling coined the term "bowl phenomenon" for this optimal arrangement of mean service times along the line.

Presumably the discovery by Hillier and Boling has the following practical implication. If the total amount of work to be performed on an item is considered a fixed quantity that can be allocated to the individual stations at the discretion of the line operator, then the operator

should deliberately unbalance the line by assigning to the interior station a workload that is less than one-third of the total load. This has the effect of decreasing the mean service time at the interior station. The unbalancing, if done in the right amount, will increase the throughput rate of the production line.

It is well known that in the case where all stations have deterministic service times the station with the greatest service time is the bottleneck that determines the line throughput rate; mean value unbalancing in this case will therefore decrease the line throughput rate. It is also known that the line throughput rate decreases with increasing service time variabilities, i.e. with increasing second moments of the service time distributions. Furthermore, the effect of service time variability on line efficiency is very different in nature from the effect of unbalance on line efficiency. It must therefore be surmised that the bowl phenomenon, as observed by Hillier and Boling, is brought about firstly by the strong dispersion of the exponential distribution and secondly by the effect of unbalancing variances which, for the exponential distribution, are necessarily coupled to the means. This is contrary to the conclusion of Hillier and Boling which attributes the bowl phenomenon only to the unbalance of the mean service times.

In order to shed light on the mechanisms by which mean value unbalance and variance unbalance affect the throughput

rate, it is more meaningful to have a method of analysis in which the mean value and the variance of each station can be specified separately. One particular question of interest is to determine the effect of unbalancing the mean service times, with the variance of each station remaining fixed, subject to the constraint that the sum of the mean service times is constant. This analysis of the bowl phenomenon is now possible through the method given in Chapter 2.

The chapter is organized as follows. In the next section we review some of the literature on the bowl phenomenon that is relevant to our work. In section 4.3 we discuss several models for constrained unbalancing. Finally, in section 4.4, we present numerical studies and the conclusions. Appendix B contains the details of the methodology of Chapter 2 as applied specifically to the model of section 4.4.

4.2. Past Work

Since the paper by Hillier and Boling in 1966, there have been many studies devoted to the investigation of the bowl phenomenon. However, in the interest of brevity we will discuss only those studies that relate to our work.

The model of constrained unbalancing of Hillier and Boling [1966] is based on a constant sum of the mean service times, with the coefficient of variation of service times at each station equal to 1.

Using the integral formulation for the throughput rate of three-station production lines derived by Muth [1973], Rao [1976] extended the analysis of the bowl phenomenon to the case where not every station has the same coefficient of variation. In his analysis he treats three different models of three-station production lines. In the first model, two stations have exponential service times and one has fixed service time. In the second model, two stations have fixed service times and one has exponential service time. In the third model, the two outside stations have exponential service times and the middle station has uniform service time. He constrains the sum of the mean service times to be constant and the coefficient of variation at each station to be constant. He finds from his numerical results that the throughput rate can be increased by assigning higher mean service times to the stations with the lower coefficient of variation of service times. He calls this optimal workload allocation "variability imbalance." For the case where the two outside stations have exponential service times and the middle stations has fixed service times, this implies that the line throughput rate can be increased by assigning higher mean service times to the middle station than to the two outside stations. The optimal arrangement of mean service times along the line is then in the form of an inverted bowl.

In some cases the effect of variability imbalance and of mean value unbalance may cancel one another as it happens

when the two outside stations have exponential service times and the middle station has uniform service time with a coefficient of variation equal to 0.5.

Hillier and Boling [1979] describe the bowl phenomenon for symmetrical production lines with up to six stations and coefficient of variation of service times equal to $1/\sqrt{m}$ where m = 1, 2, . . . , 7. They assume that the service times of all stations are Erlang with the same coefficient of variation. This choice of service time distribution lead them to the same model of constrained unbalancing as in their earlier paper [1966]. In all cases considered, the line throughput rate can be increased by assigning mean service times that decrease toward the middle of the line.

El-Rayah [1979a] considers production lines with 3, 4 and 12 stations. He treats two different cases for each model; either all stations have normal service times, or all stations have lognormal service times. He constrains the sum of the mean service times to constant and the coefficient of variation at each station to be constant. Using a simulation method, he calculates the throughput rate for four different arrangements of mean service times: a balanced line; a line with an increasing order of mean service times; a line with alternate, low and high, mean service times; and a line whose middle stations having a lower mean service time than outside stations. Using a statistical test, he compares the simulation results of

these four arrangements and draws the conclusion that the greatest throughput rate is obtained when middle stations have a lower mean service time than outside stations.

In a second paper, El-Rayah [1979b] consider the same models of production lines as in his first paper but using different constraints. He constrains the sum of the coefficient of variation of service times to be constant and the mean service time at each station to be constant and equal to 1. Using a simulation method, he calculates the throughput rate of production lines for four different arrangements of coefficient of variations: a balanced line; a line with an increasing order of coefficient of variations; a line with alternate, low and high, coefficient of variations; and a line whose middle stations having a lower coefficient of variation than outside stations. Using a statistical test, he compares the simulation result of these four arrangements and draws the conclusion that the greatest throughput rate is obtained when middle stations have a lower coefficient of variation than outside stations.

For the results of both papers, it must be noted that the accuracy attainable by simulation methods is outside the range of change in the throughput rate that is known from analytical methods. For example, in Hillier and Boling investigation the maximum percent of improvement obtained by unbalancing a four-station production line is less than 1%.

4.3. Formulations of Constrained Unbalancing

From the result of Muth [1977] as discussed in section 3.5.2, we know that the system throughput rate is affected mainly by the first two moments of the service times and that the effect of higher moments is negligible. It is therefore appropriate to base the analysis on any suitable two-parameter distribution of service times, to select the two parameters in accordance with the desired mean and variance, and then to treat the resulting throughput rate as if it were a function of the first two moments only; that is

$$r = r(\mu,\sigma)$$

where μ is a vector whose i-th element is μ_i, the mean service time of station i and σ is a vector whose i-th element is σ_i, the standard deviation of the service time of station i.

The problem that will be addressed is to establish the functional dependence of r on μ and σ. In particular, we wish to find the maximum throughput rate for the case where μ and σ are varied independently subject to certain constraints on these two parameters.

The constraints are not true physical constraints, but are rather the assumptions of a specific model. Therefore, several formulations are possible. For example, as applied to the first moment, we may select the constraint as

$$\sum_{i=1}^{K} \mu_i = C_1$$

or
$$\sum_{i=1}^{K} 1/\mu_i = C_1$$

where K is the number of stations on the line and C_1 is some positive constant. Similarly, applied to the second moment, we may select the constraint as

$$\sum_{i=1}^{K} \sigma_i = C_2$$

or
$$\sum_{i=1}^{K} \sigma_i^2 = C_2$$

or
$$\sum_{i=1}^{K} 1/\sigma_i = C_2$$

or
$$\sum_{i=1}^{K} 1/\sigma_i^2 = C_2$$

where C_2 is some positive constant.

We can formulate several models of constrained unbalancing by using different combinations of the constraints given above. In the numerical study of section 4.4 the following two constraint sets are used.

$$\sum_{i=1}^{K} \mu_i = C_1$$

$$\sum_{i=1}^{K} \sigma_i = C_2$$

$$\mu_i,\ \sigma_i \geq 0 \qquad (4\text{-}3\text{-}1)$$

and

$$\sum_{i=1}^{K} 1/\mu_i = C_1$$

$$\sum_{i=1}^{K} 1/\sigma_i = C_2$$

$$\mu_i,\ \sigma_i \geq 0 \qquad (4\text{-}3\text{-}2)$$

In comparison, the model treated by Hillier and Boling [1966, 1979], Rao [1976] or El-Rayah [1979a] has the constraint set

$$\sum_{i=1}^{K} \mu_i = C_1$$

$$\sigma_i = C_{2i} \mu_i$$

$$\mu_i \geq 0, C_{2i} \geq 0. \qquad (4\text{-}3\text{-}3)$$

In Rao's formulation the C_{2i} are not the same for all i; in Hillier and Boling's and El-Rayah's formulations the C_{2i} are the same for all i. As we can see, in formulation (4-3-3) μ and σ are not varied independently.

The model of Hillier and Boling and El-Rayah is for $C_{2i} = C_2$. When combined with the constraint $\sum_{i=1}^{K} \mu_i = C_1$, the constraint $\sigma_i = C_2\mu_i$ implies that $\sum_{i=1}^{K} \sigma_i = C_2C_1$ and thus (4-3-3) becomes a special case of the constraint set (4-3-1). In other words, the model of Hillier and Boling and El-Rayah belong to the class of models defined by (4-3-1) with the added restriction that μ and σ are coupled, instead of being selectable independently. Rao does not keep $\sum_{i=1}^{K} \sigma_i$ equal to a constant when he unbalances the mean values. As a consequence, his results are in contradiction to those of Hillier and Boling as will be shown in section 4.4.

To investigate the effect of unbalancing the mean service times alone, one should use the constraint set

$$\sum_{i=1}^{K} \mu_i = C_1$$

$$\sigma_i = C_{2i}$$

$$\mu_i \geq 0 \tag{4-3-4}$$

Similarly, to investigate the effect of unbalancing the service time variabilities alone, one should use the constraint set

$$\sum_{i=1}^{K} \sigma_i = C_2$$

$$\mu_i = C_{1i}$$

$$\mu_i \geq 0,\ C_{1i} \geq 0 \tag{4-3-5}$$

The constraint set used by El-Rayah [1979b] is the same as the constraint set (4-3-5). Note that both (4-3-4) and (4-3-5) are just special cases of (4-3-1).

It is reasonable to assume that the throughput rate in the above models has only one maximum. Hence, using the reversibility property of production lines (Muth [1979b])

the line with maximum throughput rate will be a symmetric line (see Hillier and Boling [1979] for a discussion). A symmetric line is a line with $\mu_i = \mu_{K-i+1}$ and $\sigma_i = \sigma_{K-i+1}$.

Since we are interested in finding a resource allocation that yields the maximum throughput rate, in the investigation we need to consider only the case where the line is symmetrical. By doing this, we cut the number of decision variables by more than a half and therefore we reduce the computational effort considerably.

4.4. Numerical Study

In this numerical study we treat a three-station symmetrical production line whose stations have no buffers and are not subject to breakdown. This is the simplest case for which the bowl phenomenon occurs. Station 1 and station 3 are identical; that is $\mu_1 = \mu_3$ and $\sigma_1 = \sigma_3$. After imposing constraints on the sum of the first moment and the sum of the second moment in accordance with (4-3-1) or (4-3-2) we are left with two decision variables, namely the ratios μ_1/μ_2 and σ_1/σ_2.

The throughput rate r is obtained using the methodology of Chapter 2 applied to this specific three-station case. An expression for r is given as equation (B-14) in Appendix B. Using this expression, the throughput rate r can be computed for arbitrary distributions of the service time of the middle station and for special phase type distributions of the service times of the two end stations.

A computer program that uses this method was written. This program computes the throughput rate r over a specified range of values of the two parameters $x = \mu_1/\mu_2$ and $y = \sigma_1/\sigma_2$. Several cases are investigated. These cases are shown in the tables below.

Table 4.1. Case 1.a.

	Station 1	Station 2	Station 3
service time distribution	Erlang	deterministic	Erlang
μ_i (balanced condition)	1	1	1
σ_i (balanced condition)	0.8	0	0.8

Constraints: The sum of the mean values is constant.
The variance of each station is constant.

Table 4.2. Case 1.b. (Rao's Formulation)

	Station 1	Station 2	Station 3
service time distribution	exponential	deterministic	exponential
μ_i (balanced condition)	1	1	1
σ_i (balanced condition)	1	0	1

Constraints: The sum of the mean values is constant.
The coefficient of variation of each station is constant.

Table 4.3. Case 2.

	Station 1	Station 2	Station 3
service time distribution	Erlang	Erlang	Erlang
μ_i (balanced condition)	1	1	1
σ_i (balanced condition)	0.8	0.8	0.8

Constraints: The sum of the mean values is constant.
The sum of the standard deviations is constant.

Table 4.4. Case 3.

	Station 1	Station 2	Station 3
service time distribution	Erlang	Erlang	Erlang
μ_i (balanced condition)	1	1	1
σ_i (balanced condition)	0.8	0.8	0.8

Constraints: The sum of the reciprocal mean values is constant.
The sum of the reciprocal standard deviations is constant.

The results of case 1.a and case 1.b are shown in figure 4.1 where the relative throughput rate is plotted as a function of the ratio μ_1/μ_2. The relative throughput rate, r_R, is defined as r/r_b where r_b is the throughput rate of the balanced line. From this figure we can see that if the variance of each station is kept constant while unbalancing the mean service times (case 1.a) then the bowl phenomenon occurs, i.e. the maximum throughput rate is obtained by assigning a lower mean service time to the middle station. This result contradicts the variability imbalance of Rao, which says that the middle station should have a higher mean service time (case 1.b). Since in Rao's

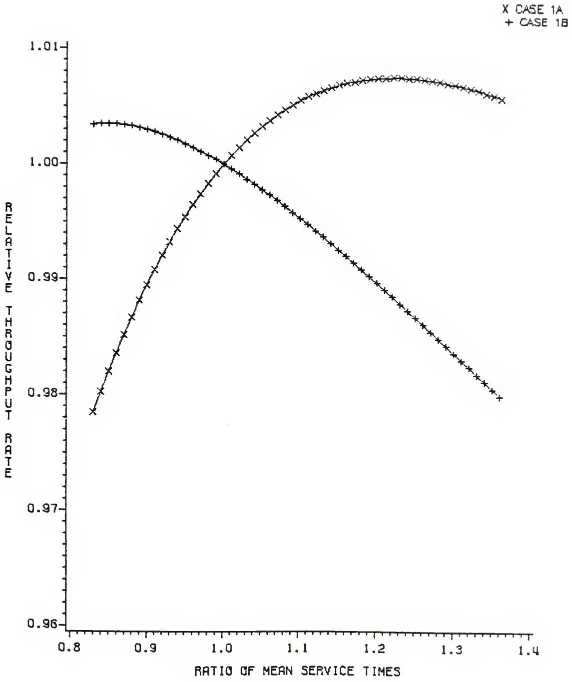


Figure 4.1. Comparison of case 1A and case 1B.

formulation the variance of each station is not kept constant, it can be concluded that his result is due mainly to the change in the variances of service times that is coupled to the mean service times. This conclusion can be explained as follows. In Rao's formulation, as the two outside stations are made faster their variances are also made less. Since the variance of the middle station is always zero, the total variance is also made less, and therefore one has a better production line. However, as the two outside stations become faster, the effect of blocking and starving becomes more severe and makes the throughput rate decrease. For this reason, the line should be unbalanced in the fashion of an inverted bowl.

Figure 4.2 shows contours of constant throughput rate obtained for case 2 computed using SAS package. The values associated with the contours are the relative throughput rate, which is equal to one for the balanced case. From this figure we can draw some conclusions. First, when $y = \sigma_1/\sigma_2$ is kept equal to 1, i.e. $\sigma_1 = \sigma_2 = \sigma_3 = 0.8$ and μ are unbalanced, the maximum improvement over the balanced condition is 0.4% and occurs at $x = \mu_1/\mu_2 = 1.18$. As a comparison Hillier and Boling obtained an improvement of about .52% at $x = \mu_1/\mu_2 = 1.27$ and, therefore, $y = \sigma_1/\sigma_2 = 1.27$. This tells us that the bowl phenomenon is more pronounced when the variabilities are unbalanced together with the mean values. We see from the contour plot that the maximum improvement is about .55% and occurs at $\mu_1/\mu_2 = 1.27$

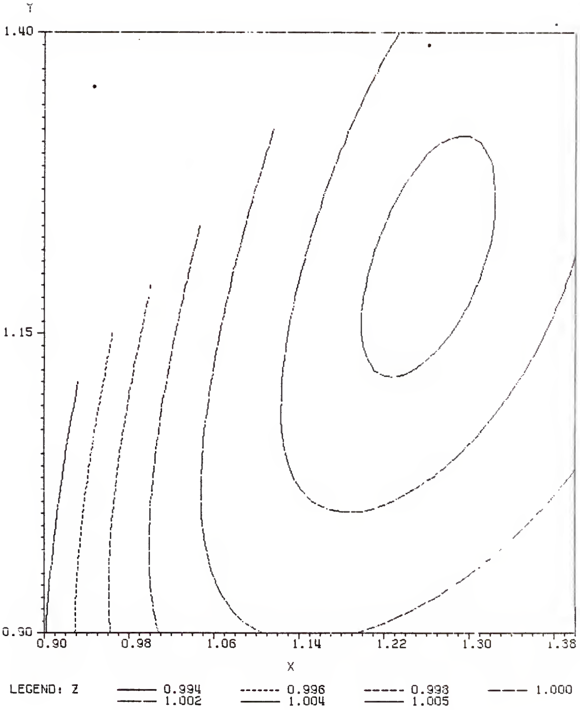


Figure 4.2. Result of case 2.

and $\sigma_1/\sigma_2 = 1.23$. Secondly, in order to get the highest throughput rate we should assign less work (lower mean value) to the more consistent (lower variability) stations. This trend is in line with the result of case 1.a, but contradicts the result of case 1.b. Thirdly, there are points in the contours at which the throughput rate can be increased by unbalancing the mean service times, but is insensitive to unbalancing of the service time variabilities. Then there are other points at which the throughput rate can be increased by unbalancing the service time variabilities but is insensitive to unbalancing of the mean service times. In other words, there are points at which the gradient of the throughput rate is in the direction of the x-axis, and there are points at which the gradient is in the direction of the y-axis.

Contours of constant throughput rate obtained for case 3 are given in figure 4.3. This figure is very similar to figure 4.2 for case 2 except that the maximum improvement is .21% and occurs at $x = \mu_1/\mu_2 = 1.09$ and $y = \sigma_1/\sigma_2 = 1.02$.

In figure 4.4 and 4.5 we show contours of constant throughput rate for case 2 and case 3 applied to a paced production line. We can see from these figures that unbalancing a paced production line in any direction decreases the throughput rate.

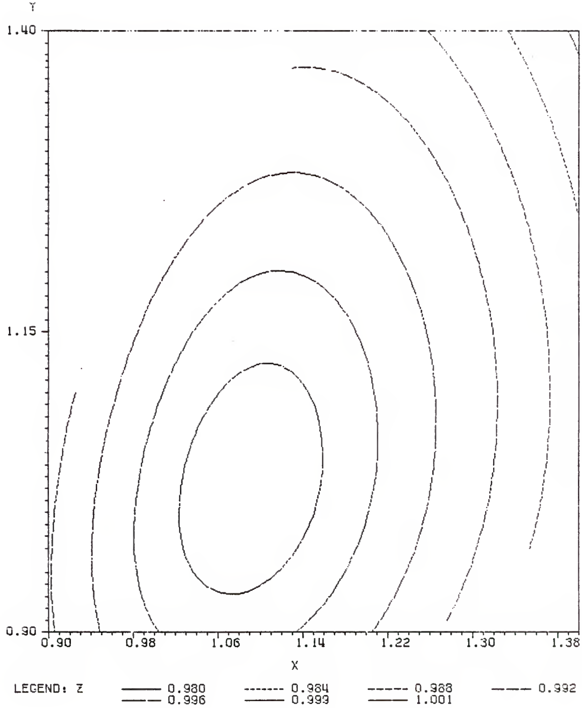


Figure 4.3. Result of case 3.

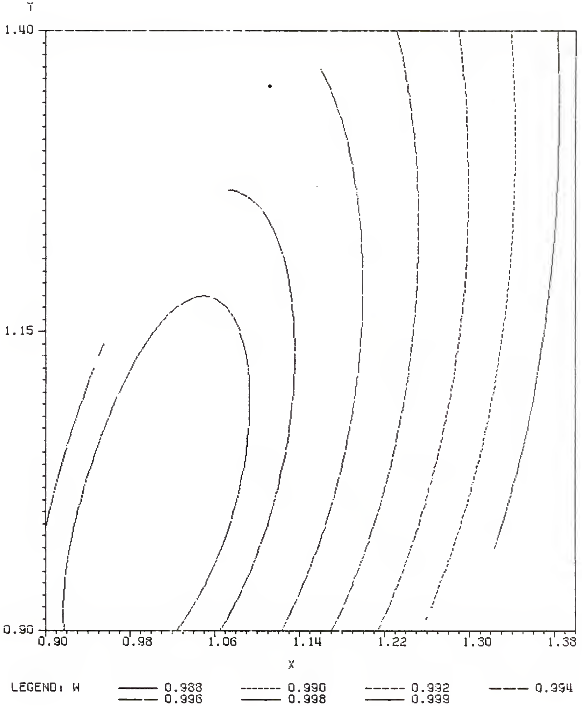


Figure 4.4. Result of case 2 in a paced production line.

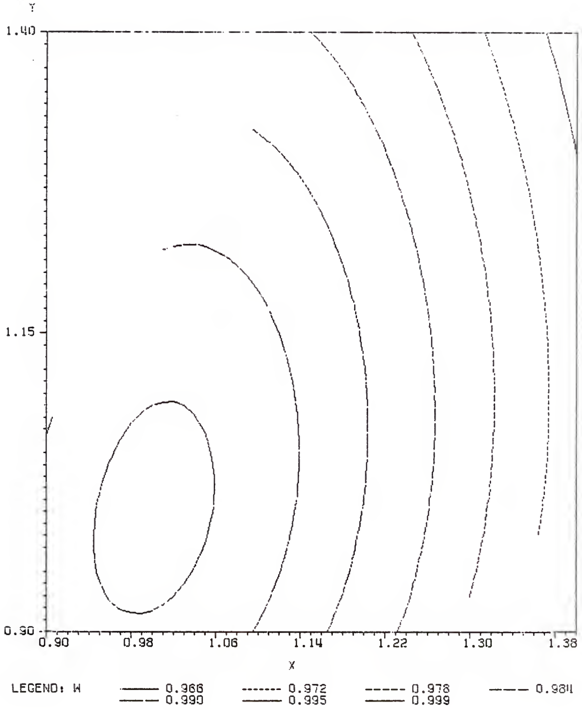


Figure 4.5. Result of case 3 in a paced production line.

CHAPTER 5

SUMMARY

The research presented in this dissertation was undertaken because of the paucity of numerical methods that are applicable to manufacturing systems in general and production lines in particular. The objective was twofold: first, to develop an efficient solution procedure for the throughput rate of a production line with an arbitrary number of stations that can handle a more general distributions of service time, repair time and time between breakdown than the available methods; secondly, to use this procedure to investigate the effect of changing station parameter on the line's throughput rate.

Since in the Markov model the distribution of service time is restricted to exponential distribution, the fundamental model used in this dissertation is a holding time model. This approach can be applied to production lines with an arbitrary number of stations and an arbitrary distribution of service time.

Following is a list of the accomplishments of this dissertation in the order of their importance.

1. We developed a solution procedure for the throughput rate that is more efficient and can handle a class of

service time distributions that is more general than previous methods. We call this class of distributions special phase type distribution.

2. We obtained a closed form expression for the throughput rate of a three-station production line that is more general than expressions heretofore available.
3. We extended the holding time model to incorporate breakdown and repair. This was accomplished through an equivalent service time, called virtual service time, that can be used in a production line without station breakdown. We obtained a general formula that relates the distribution of virtual service time to the distributions of service time, repair time, and time between breakdowns.

Using these results, we have been able to do the following:

1. We calculated the throughput rate of K-station production lines without station breakdown for K = 3, 4, . . ., 10 with general Erlang service time. The most complex case for which a solution was previously given in the literature is K = 6 with exponential service time.
2. We calculated the throughput rate of production lines without station breakdown whose service time distributions are in the class of special phase type that is more general than Erlang distribution. Most analysis of production lines previously carried by assuming special Erlang service time.

3. We developed a method to calculate the throughput rates of production lines with station breakdown for the case where the distributions of service time, repair time and time between breakdowns are not exponential. Previously, only the case where all three distributions are exponential could be handled.
4. We produced an empirical formula for the throughput rate of balanced production lines.
5. We developed a method of parametric analysis for the throughput rate of production lines that uses the mean values and the variances of service times as the input. Using this analysis we produced sets of contours of constant throughput rate of three-station production lines. These contours can be used to separate the effect of mean value unbalance from that of variability unbalance on the throughput rate.

The following open problems are left as areas of future research:

1. Solving the question of whether or not $H_j(n-1)$ and $I_{j-1}(n)$ are statistically independent in a more definitive form.
2. Extending the current solution procedure in the following ways:
   - Generalize the procedure so that it can handle service time distribution with repeated and/or complex conjugate poles. This case was excluded to

simplify the derivation of the equations used in the procedure.

- Develop a procedure that includes buffers in the analysis. The current procedure treats buffers as stations with zero service time. While it is possible to handle buffers this way, the procedure becomes very inefficient if buffers have a large capacity.
- Exploit the structure of the resulting equations so that they can be solved more efficiently. Preliminary investigation shows that the system of equations that need to be solved has upper triangular structure. This structure suggests a better procedure to solve this system of equations than the procedure currently used.

3. Producing an empirical formula that relates the throughput rate to the mean and variance of each station. Special form of such a formula is given in Chapter 2 for the case where all stations have the same mean value and variance.

APPENDIX A

PROBABILITY DISTRIBUTION OF SPECIAL PHASE TYPE

Consider a Markov process with one absorbing state. Let state i, i = i,2. . . . ,m be the transient states of this process and state m + 1 be the absorbing state. Such a process has an (m+1)x(m+1) transition rate matrix of the following form:

$$A = \begin{pmatrix} C & c \\ 0 & 0 \end{pmatrix} \tag{A-1}$$

where the m x m submatrix C satisfies the following properties:

$$c_{ii} < 0$$

$$c_{ij} \geq 0 \qquad \text{for } i \neq j$$

$$\sum_{j=1}^{m} c_{ij} \leq 0$$

Also

$$Cu + c = 0$$

where u is a vector with all elements equal to 1.

If this process starts out in state i with probability q_i, i = 1, 2, . . ., m + 1, then the distribution of time to absorbtion into state m + 1 is given as

$$F(t) = \begin{cases} 1 - q\, e^{Ct} u & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A-2}$$

where q is a vector with q_i, i = 1, 2, . . ., m as its i-th element.

The probability distribution of phase type is defined by Neuts [1981] as the probability of the time to absorbtion in a Markov process with one absorbing state. Therefore, in general, this probability distribution can always be written in the form (A-2). The Laplace-Stieltjes transform of (A-2) is

$$\begin{aligned} F^*(s) &= q_{m+1} + q\,(sI-C)^{-1}(-C)u \\ &= q_{m+1} + q\,(sI-C)^{-1} c \end{aligned} \tag{A-3}$$

Considering equation (A-3), the probability distribution of phase type has the property that its Laplace-Stieltjes transform is a rational function whose poles are in the left hand plane. Conversely, a probability distribution whose Laplace-Stieltjes transform is a rational function is of phase type that can always be written in the form (A-2). This class of probability distributions is a proper subset

of the class of Coxian distributions introduced by Cox [1955].

A subclass of the class of phase type distributions comprises those distribution functions whose Laplace-Stieltjes transforms have only real and distinct poles. We call these distributions special phase type distributions. If F(t) is a special phase type distribution, we can write F(t) as

$$F(t) = \begin{cases} 1 - \sum_{i=1}^{m} a_i e^{-b_i t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(A-4)}$$

where $b_i > 0$, $0 \leq \sum_{i=1}^{m} a_i \leq 1$ and $m \geq 1$, and where the parameters a_i are restricted to values that make F(t) a valid distribution function. The relation between the parameters in (A-4) and (A-2) is obtained by comparing the Laplace-Stieltjes transforms of both expressions. As a result, we have

$$q_{m+1} = 1 - \sum_{i=1}^{m} a_i$$

$-b_i$ is the i-th eigenvalue of C

$$a_i = \lim_{s \to -b_i} (s+b_i)\ q\ (sI-C)^{-1}\ c$$

The class of special phase type distribution is quite general. It includes general Erlang and hyperexponential distributions as special cases. The restriction to real and distinct poles is important for the development of simple numerical solution procedures for the problem discussed in this study. Several properties of special phase type distributions are stated as theorems and lemmas in what follows.

Definition: $F_X(t)$ is a special phase type distribution if and only if it can be expressed in the form (A-4).

From this definition it follows that X must be a nonnegative random variable.

Theorem A.1: Let Z = max [X,Y]. If $F_X(t)$ and $F_Y(t)$ are special phase type distributions and X is statistically independent of Y, then $F_Z(t)$ is also a special phase type distribution.

Proof: Let $F_X(t) = 1 - \sum_{i=1}^{m} a_i \, e^{-b_i t}$ for $t \geq 0$

$$F_Y(t) = 1 - \sum_{i=1}^{n} c_i \, e^{-d_i t} \quad \text{for } t \geq 0$$

Then $F_Z(t) = F_X(t) \, F_Y(t)$

$$= 1 - \sum_{i=1}^{m} a_i \, e^{-b_i t} - \sum_{i=1}^{n} c_i \, e^{-d_i t}$$

$$+ \left(\sum_{i=1}^{m} a_i \, e^{-b_i t}\right)\left(\sum_{i=1}^{n} c_i \, e^{-d_i t}\right)$$

which has a rational Laplace-Stieltjes transform whose poles are all distinct, real and negative.

Theorem A.2: Let $Z = \max [X-Y,0]$. If $F_X(t)$ is a special phase type distribution and Y is a nonnegative random variable independent of X, then $F_Z(t)$ is also a special phase type distribution.

Proof:

$$F_Z(t) = 1 - \int_{x=0}^{\infty} F_Y(x)\, dF_X(x+t) \quad \text{for } t \geq 0$$

$$= 1 - \sum_{i=1}^{m} a_i \left(\int_{x=0}^{\infty} F_Y(x) b_i e^{-b_i x} dx \right) e^{-b_i t}$$

$$= 1 - \sum_{i=1}^{m} a_i\, g_i\, e^{-b_i t}$$

which is a special phase type distribution.

Consider the following recursive relationships which are given as equation (2-3-1), (2-3-6) and (2-3-7) in Chapter 2.

$$R_j^+(n) = \max [H_j(n) - I_{j-1}(n+1),\ 0] \qquad \text{(A-5)}$$

$$H_j(n) = \max [S_j(n),\ R_{j+1}^+(n-1)] \qquad \text{(A-6)}$$

$$H_j(n) = S_j(n) + B_j(n) \qquad \text{(A-7)}$$

$$R_{K+1}(n) = 0$$

where $S_j(n)$, $j\varepsilon\kappa$ are random variables having special phase type distributions. We have the following lemmas.

<u>Lemma A.1</u> $F_{R_j^+}(t)$ and $F_{H_j}(t)$ for all $j\varepsilon\kappa$ are special phase type distributions.

<u>Proof</u>: Since $R_{K+1}(n) = 0$, $F_{R_{K+1}^+}(t) = u(t)$ which is a special phase type distribution. Therefore, from equation (A-6) for j=K and theorem A.1, $F_{H_K}(t)$ is a special phase type distribution. Using this result together with equation (A-5) for j = K and theorem A.2, $F_{R_K^+}(t)$ is also a special phase type distribution. Continuing in this fashion, we can show that, for all $j\varepsilon\kappa$, $F_{R_j^+}(t)$ and $F_{H_j}(t)$ are special phase type distributions.

<u>Lemma A.2</u>: $F_{B_j}(t)$ is a special phase type distribution.

<u>Proof</u>: From equations (A-6) and (A-7) we have

$$B_j(n) = \max\ [R_{j+1}^+(n-1) - S_j(n),\ 0]$$

Therefore, by theorem A.2, $F_{B_j}(t)$ is a special phase type distribution.

Consider the recursive relationships

$$I_j(n) = \max\ [S_{j-1}(n) - R_j(n-1),\ 0] \tag{A-8}$$

$$R_j(n) = H_j(n) - I_{j-1}(n+1) \tag{A.9}$$

which are given as equation (2-3-6) and (2-3-8) in Chapter 2. Since $R_j(n)$ is not a nonnegative random variable, theorem A.2 is not applicable to equation (A-8). Therefore, we cannot say, in general, that $F_{I_j}(t)$ will have a special phase type distribution. It turns out that, in some cases, the Laplace-Stieltjes transform of $F_{I_j}(t)$ has repeated poles. However, in the approximation method we use the following relationship

$$\hat{I}_j(n) = \max\ [S_{j-1}(n) - R_j^+(n-1),\ 0] \tag{A-10}$$

which is given as equation (2-6-27) in Chapter 2 to approximate the true value of $I_j(n)$. We have the following lemma.

<u>Lemma A.3</u>: $F_{\hat{I}_j}(t)$ is a special phase type distribution.

Proof: Since $F_{S_{j-1}}(t)$ is a special phase type distribution, the proof follows from theorem A.2 and equation (A-10).

These three lemmas show that if $F_{S_j}(t)$ for all $j \epsilon \kappa$ are special phase type distributions, then $F_{R_j^+}(t)$, $F_{H_j}(t)$, $F_{B_j}(t)$ and $F_{\hat{I}_j}(t)$ are special phase type distributions, but $F_{I_j}(t)$ is not. Since $F_{\hat{I}_j}(t)$ is used in the approximation method as opposed to $F_{I_j}(t)$ in the exact method, these lemmas explain why the approximation method is simpler and more desirable than the exact method.

APPENDIX B

A FORMULA FOR THE THROUGHPUT RATE OF A THREE-STATION PRODUCTION LINE

The system of integral equations for the holding time model of a three-station production line has a special structure. For the case when the service time distributions of the two outside stations are of special phase type this system of integral equations can be transformed into a system of linear algebraic equations with the number of equations equal to the number of stages of the service time distribution of the last station. Furthermore, closed form expressions for the throughput rate can also be obtained for special service time distributions of the last station.

When K = 3, equations (2-4-4) through (2-4-6) reduce to

$$F_{I_2}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{S_2}(x)\, F_{R_3^+}(x)\, dF_{S_1}(x+t) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(B-1)}$$

$$F_{R_3^+}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_2}(x)\, dF_{S_3}(x+t) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(B-2)}$$

Let the service time distributions of station 1 and 3 be of special phase type, i.e.

$$F_{S_1}(t) = 1 - \sum_{i=1}^{m} \beta_i e^{-\alpha_i t} \qquad \text{for } t \geq 0 \qquad \text{(B-3)}$$

and

$$F_{S_3}(t) = 1 - \sum_{j=1}^{n} \delta_j e^{-\gamma_j t} \qquad \text{for } t \geq 0 \qquad \text{(B-4)}$$

where m and n are the number of stages of these distributions.

Substituting (B-4) into (B-2) yields

$$F_{R_3^+}(t) = 1 - \sum_{j=1}^{n} \left(\int_0^\infty F_{I_2}(x)\, \delta_j \gamma_j e^{-\gamma_j x} dx \right) e^{-\gamma_j t}$$

$$= 1 - \sum_{j=1}^{n} a_j e^{-\gamma_j t} \qquad \text{for } t \geq 0 \qquad \text{(B-5)}$$

where

$$a_j = \int_0^\infty F_{I_2}(x)\, \gamma_j \delta_j e^{-\gamma_j x} dx \qquad \text{for } j = 1, 2, \ldots, n$$

(B-6)

Similarly, substituting (B-3) into (B-1) yields

$$F_{I_2}(t) = 1 - \sum_{i=1}^{m} \left(\int_0^{\infty} F_{S_2}(x)\, F_{R_3^+}(x)\, \beta_i \alpha_i e^{-\alpha_i x} dx \right) e^{-\alpha_i t}$$

$$= 1 - \sum_{i=1}^{m} b_i e^{-\alpha_i t} \qquad \text{for } t \geq 0 \qquad \text{(B-7)}$$

where

$$b_i = \int_0^{\infty} F_{S_2}(x)\, F_{R_3^+}(x)\, \beta_i \alpha_i e^{-\alpha_i x} dx$$

$$\text{for } i = 1, 2, \ldots, m \qquad \text{(B-8)}$$

Substituting (B-7) into (B-6) yields

$$a_j = \delta_j - \sum_{i=1}^{m} b_i \gamma_j \delta_j \,/\, (\alpha_i + \gamma_j)$$

$$\text{for } j = 1, 2, \ldots, m \qquad \text{(B-9)}$$

and substituting (B-5) into (B-8) yields

$$b_i = \int_0^{\infty} F_{S_2}(x) \left(1 - \sum_{j=1}^{n} a_j e^{-\gamma_j x}\right) \beta_i \alpha_i e^{-\alpha_i x} dx$$

$$= \beta_i \alpha_i \bar{F}_{S_2}(\alpha_i) - \sum_{j=1}^{n} a_j \beta_i \alpha_i \bar{F}_{S_2}(\alpha_i + \gamma_j)$$

$$\text{for } i = 1, 2, \ldots, m \qquad \text{(B-10)}$$

where $\bar{F}_{S_2}(s)$ is the Laplace transform of $F_{S_2}(t)$.

To put equations (B-9) and (B-10) in a more compact form, we define the column vectors

$$a = [a_1, a_2, \ldots, a_n]^T$$
$$b = [b_1, b_2, \ldots, b_m]^T$$
$$c = [c_1, c_2, \ldots, c_m]^T$$

where $c_i = \beta_i \alpha_i F_{S_2}(\alpha_i)$. We further define the matrices

$$P = [p_{ij}]_{mxn}$$

$$Q = [q_{ij}]_{nxm}$$

where

$$p_{ij} = \beta_i \alpha_i \bar{F}_{S_2}(\alpha_i + \gamma_j)$$

$$q_{ij} = \gamma_j \delta_j / (\alpha_i + \gamma_j)$$

Equations (B-9) and (B-10) becomes

$$a = \delta - Qb \qquad \text{(B-11)}$$

$$b = c - Pa \qquad \text{(B-12)}$$

where δ is a column vector whose j-th element is δ_j.

Eliminating b from these two equations yields a system of equations defining the vector a

$$a = \delta - Q\,(c - Pa)$$

or

$$a = (I - QP)^{-1}\,(\delta - Qc) \tag{B-13}$$

Given the parameters α_i, β_i, γ_j, δ_j and the Laplace transform of the service time distribution of the middle station, the value of a is calculated with equation (B-13). Once this value has been found, the mean interdemand time $E[H_1]$ is calculated by substituting (B-5) into (2-5-5) and then (2-5-3); namely

$$E[H_1] = \int_0^\infty (1 - F_{S_1}(t)\, F_{S_2}(t)\, F_{R_3^+}(t))\,dt$$

$$= \int_0^\infty (1 - F_{S_1}(t)\, F_{S_2}(t)\, (1 - \sum_{j=1}^{n} a_j e^{-\gamma_j t}))\,dt$$

Substituting (B-3) for $F_{S_1}(t)$ in the above equation yields

$$E[H_1] = \int_0^{\infty} (1 - F_{S_2}(t))dt + \sum_{i=1}^{m} \beta_i \int_0^{\infty} F_{S_2}(t)e^{-\alpha_i t}dt$$

$$+ \sum_{j=1}^{n} a_j \left(\int_0^{\infty} F_{S_2}(t) \; (1 - \sum_{i=1}^{m} \beta_i e^{-\alpha_i t})e^{-\gamma_j t} dt\right)$$

$$= \mu_2 + \sum_{i=1}^{m} \beta_i \bar{F}_{S_2}(\alpha_i) +$$

$$\sum_{j=1}^{n} a_j (\bar{F}_{S_2}(\gamma_j) - \sum_{i=1}^{m} \beta_i \bar{F}_{S_2}(\alpha_i + \gamma_j)) \tag{B-14}$$

where μ_2 is the mean service times of the middle station.

From the above derivation we can see that formula (B-14) can be used for arbitrary service time distribution of the middle station as long as the Laplace transform of this distribution function is known. For example, if the middle station has a deterministic service time equal to μ_2, then $\bar{F}_{S_2}(s) = e^{-\mu_2 s}/s$.

Equation (B-14) is not a closed form expression since in order to find the value of a we must invert the nxn matrix (I-QP). In general this inversion must be done numerically. However, for some special cases, the value of

a is readily available from equation (B-13). These special cases are given below

<u>Case 1: m = n = 1</u>

This case means that the two outside stations have exponential service times. Therefore

$$\beta_i = \delta_j = 1 \; ; \; \alpha_i = \alpha \text{ and } \gamma_j = \gamma$$

Equation (B-14) reduces to

$$E[H_1] = \mu_2 + \bar{F}_{S_2}(\alpha) + a\,(\bar{F}_{S_2}(\gamma) - \bar{F}_{S_2}(\alpha + \gamma)) \qquad \text{(B-15)}$$

where a is obtained from equation (B-13) as follows

$$a = (1 - qc) \,/\, (1 - qp)$$

$$c = \alpha\, \bar{F}_{S_2}(\alpha)$$

$$p = \alpha\, \bar{F}_{S_2}(\alpha + \gamma)$$

$$q = \gamma \,/\, (\alpha + \gamma)$$

This closed form expression is more general than the ones given by Hunt [1956], Rao [1976] or Muth [1984] in the sense that the middle station can have arbitrary service

time distribution. While Rao [1976] derived two different formulas for the middle station having deterministic and uniform service times, and Muth [1984] derived a formula for the middle station having exponential service time, these three cases are represented here by the general formula given by (B-15).

Case 2: $n = 1$

In this case only the last station has exponential service time. Therefore

$$\delta_j = 1 \quad \text{and} \quad \gamma_j = \gamma$$

Equation (B-14) reduces to

$$E[H_1] = \mu_2 + \sum_{i=1}^{m} \beta_i \bar{F}_{S_2}(\alpha_i) +$$

$$a \left(\bar{F}_{S_2}(\gamma) - \sum_{i=1}^{m} \beta_i \bar{F}_{S_2}(\alpha_i + \gamma)\right) \qquad \text{(B-16)}$$

where

$$a = \left(1 - \sum_{i=1}^{m} q_i c_i\right) / \left(1 - \sum_{i=1}^{m} q_i p_i\right)$$

$$c_i = \alpha_i \beta_i \bar{F}_{S_2}(\alpha_i)$$

$$p_i = \alpha_i \beta_i \bar{F}_{S_2}(\alpha_i + \gamma)$$

$$q_i = \gamma / (\alpha_i + \gamma)$$

<u>Case 3: n = 2</u>

Without going into details of the derivations, for this case we obtain from equation (B-13)

$$a_1 = \frac{1}{\Delta} \left(1 - \sum_{i=1}^{m} q_{2i} p_{i2}\right) \left(\delta_1 - \sum_{i=1}^{m} q_{1i} c_i\right)$$

$$+ \frac{1}{\Delta} \left(\sum_{i=1}^{m} q_{1i} p_{i2}\right) \left(\delta_2 - \sum_{i=1}^{m} q_{2i} c_i\right)$$

$$a_2 = \frac{1}{\Delta} \left(1 - \sum_{i=1}^{m} q_{1i} p_{i1}\right) \left(\delta_2 - \sum_{i=1}^{m} q_{2i} c_i\right)$$

$$+ \frac{1}{\Delta} \left(\sum_{i=1}^{m} q_{2i} p_{i1}\right) \left(\delta_1 - \sum_{i=1}^{m} q_{1i} c_i\right)$$

where

$$\Delta = (1 - \sum_{i=1}^{m} q_{1i}p_{i1})\ (1 - \sum_{i=1}^{m} q_{2i}p_{i2})$$

$$- (\sum_{i=1}^{m} q_{1i}p_{i2})\ (\sum_{i=1}^{m} q_{2k}p_{i2})$$

c_i, p_{ij} and q_{ij} are as given previously. The mean interdemand time is calculated using equation (B-14) with a_1 and a_2 as given above.

APPENDIX C

THE DERIVATION OF $F_{I_j}(t)$ AND $F_{R_j}(t)$

We derive the formula for the distribution of a random variable that is a function of the difference of two independent random variables. Let X and Y be two independent random variables with distribution functions $F_X(t)$ and $F_Y(t)$. The distribution function of X - Y can be derived using the law of total probability. By conditioning on X = x, we obtain

$$F_{X-Y}(t) = P[X-Y \leq t]$$

$$= \int_{x=-\infty}^{\infty} P[x - Y \leq t] \, d\, F_X(x)$$

$$= \int_{x=-\infty}^{\infty} (1 - F_Y(x-t)) \, d\, F_X(x)$$

$$= 1 - \int_{x=-\infty}^{\infty} F_Y(x-t) \, d\, F_X(x) \qquad \text{(C-1)}$$

If X is a nonnegative random variable, then equation (C-1) reduces to

$$F_{X-Y}(t) = 1 - \int_{x=0}^{\infty} F_Y(x-t)\ d\ F_X(x) \tag{C-2}$$

By letting x-t = y, equation (C-2) becomes

$$F_{X-Y}(t) = 1 - \int_{y=-t}^{\infty} F_Y(y)\ d\ F_X(y + t) \tag{C-3}$$

If Y is also a nonnegative random variable, then equation (C-2) reduces to

$$F_{X-Y}(t) = \begin{cases} 1 - \int_{x=t}^{\infty} F_Y(x-t)\ d\ F_X(x) & \text{for } t \geq 0 \\ 1 - \int_{x=0}^{\infty} F_Y(x-t)\ d\ F_X(x) & \text{for } t < 0 \end{cases} \tag{C-4}$$

and equation (C-3) reduces to

$$F_{X-Y}(t) = \begin{cases} 1 - \int_{y=0}^{\infty} F_Y(y)\ d\ F_X(y + t) & \text{for } t \geq 0 \\ 1 - \int_{y=-t}^{\infty} F_Y(y)\ d\ F_X(y + t) & \text{for } t < 0 \end{cases} \tag{C-5}$$

Let the random variable Z be defined as $Z = \max[X-Y, 0]$ where X is a nonnegative random variable, then the distribution function of Z is obtained as

$$F_Z(t) = \begin{cases} 1 - \int_{y=-t}^{\infty} F_Y(y)\, d\, F_X(y + t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{C-6}$$

Consider the random variable R_j that is computed using relationship $R_j(n) = H_j(n) - I_{j-1}(n+1)$ given as equation (2-3-6) in Chapter 2, where $H_j(n)$ and $I_{j-1}(n+1)$ are both nonnegative random variables. From equations (C-4) and (C-5) we obtain

$$F_{R_j}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x)\, d\, F_{H_j}(x + t) & \text{for } t \geq 0 \\ 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t)\, d\, F_{H_j}(x) & \text{for } t < 0 \end{cases} \tag{C-7}$$

Since $R_j^+(n) = \max[R_j(n), 0]$ and $R_j^-(n) = \min[R_j(n), 0]$, we have

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x)\, d\, F_{H_j}(x + t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{C-8}$$

and

$$F_{R_j^-}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t)\, d\, F_{H_j}(x) & \text{for } t \leq 0 \\ 0 & \text{for } t > 0 \end{cases} \tag{C-9}$$

Similarly, the distribution function of I_j is obtained from the relationship $I_j(n) = \max[S_{j-1}(n) - R_j(n-1), 0]$ given as equation (2-3-8) in Chapter 2, where S_{j-1} is a nonnegative random variable. From equation (C-6) we have

$$F_{I_j}(t) = \begin{cases} 1 - \int_{x=-t}^{\infty} F_{R_j}(x)\, d\, F_{S_{j-1}}(x + t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or

$$F_{I_j}(t) = \begin{cases} 1 - \int_{x=-t}^{\infty} F_{R_j^-}(x)\,dF_{S_{j-1}}(x+t) - \int_{x=0}^{\infty} F_{R_j^+}(x)\,dF_{S_{j-1}}(x+t) \,. & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{C-10}$$

BIBLIOGRAPHY

Altiok, T. 1982. Approximate Analysis of Exponential Tandem Queues with Blocking. European Journal of Operational Research, 11, 390-397.

Altiok, T., and S. Stidham Jr. 1983. The Allocation of Interstage Buffer Capacities in Production Lines. IIE Transactions, 15, 292-299.

Anderson D.R., and D.C. Moodie. 1969. Optimal Buffer Storage Capacity in Production Line Systems. International Journal of Production Research, 7, 233-240.

Avi-Itzhak, B., and M. Yadin. 1965. A sequence of Two Servers with No Intermediate Queue. Management Science, 11, 553-564.

Berman, O. 1982. Efficiency and Production Rate of a Transfer Line with Two Machines and a Finite Storage Buffer. European Journal of Operational Research, 9, 295-308.

Buzacott, J.A. 1967. Automatic Transfer Lines with Buffer Stocks. International Journal of Production Research, 5, 183-200.

Buzacott, J.A. 1968. Prediction and Efficiency of Production Systems without Internal Storage. International Journal of Production Research, 6, 173-188.

Buzacott, J.A. 1972. The Effect of Station Breakdowns and Random Processing Times on the Capacity of Flow Lines with In-Process Storage. AIIE Transactions, 4, 308-312.

Buzacott, J.A., and L.E. Hanifin. 1978. Models of Automatic Transfer Lines with Inventory Banks - A Review and Comparison. AIIE Transactions, 10, 197-207.

Cox, D.R. 1955. A Use of Complex Probabilities in the Theory of Stochastic Processes. Proc. Cambridge Philosophical Society, 51, 313-319.

Cox, D.R. 1962. Renewal Theory. Methuen, London.

Dorf, R.C. 1967. Modern Control Systems. Addison-Wesley, Reading MA.

El-Rayah, T.E. 1979a. The efficiency of Balanced and Unbalanced Production Lines. International Journal of Production Research, 17, 61-75.

El-Rayah, T.E. 1979a. The Effect of Inequality of Interstage Buffer Capacities and Operation Times Variability on the Efficiency of Production Line Systems. International Journal of Production Research, 17, 77-89.

Freeman, M.C. 1964. The Effects of Breakdowns and Interstage Storage on Production Line Capacity. Journal of Industrial Engineering, 15, 194-200.

Gaver, D.P. Jr. 1962. A Waiting Line with Interrupted Service, Including Priorities. Journal of the Royal Statistical Society-Series B, 24, 73-90.

Gershwin, S.B., and O. Berman. 1981. Analysis of Transfer-Lines Consisting of Two Unreliable Machines with Random Processing Times and a Finite Storage Buffer. AIIE Transactions, 13, 2-11.

Gershwin, S.B., and I.C. Schick. 1980. Continuous Model of an Unreliable Two-Stage Material Flow System With a Finite Interstage Buffer. Report LIDS-R-1039, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology.

Gershwin, S.B., and I.C. Schick. 1983. Modeling and Analysis of Three-Stage Transfer Lines with Unreliable Machines and Finite Buffers. Operations Research, 31, 354-380.

Hillier, F.S., and R.W. Boling. 1966. The Effect of Some Design Factors on the Efficiency of Production Lines with Variable Operation Times. Journal of Industrial Engineering, 17, 651-658.

Hillier, F.S., & R.W. Boling. 1967. Finite Queues in Series with Exponential or Erlang Service Times: A Numerical Approach. Operations Research, 15, 286-303.

Hillier, F.S., and R.W. Boling. 1979. On the Optimal Allocation of Work in Symmetrically Unbalanced Production Line Systems with Variable Operation Times. Management Science, 25, 721-728

Ho, Y.C., M.A. Eyler and T.T. Chien. 1979. A Gradient Technique for General Buffer Storage Design in a Production Line. International Journal of Production Research, 17, 557-580.

Hunt, G.C. 1956. Sequential Arrays of Waiting Lines. Operations Research, 4, 674-684.

Ignall, E., and A.M. Silver. 1977. The Output of a Two-Stage System with Unreliable Machines and Limited Storage. AIIE Transactions, 9, 183-188.

Masso, J., and M.L. Smith. 1974. Interstage Storage for Three Stage Lines Subject to Stochastic Failures. AIIE Transactions, 6, 354-358.

Murphy, R.A. 1978. Estimating the Output of a Series Production System. AIIE Transactions, 10, 139-148.

Muth, E.J. 1970. Excess Time, A Measure of System Repairability. IEEE Transactions on Reliability, 19, 16-19.

Muth, E.J. 1973. The Production Rate of a Series of work Stations with Variable Service Times. International Journal of Production Research, 11, 155-159.

Muth, E.J. 1977. Numerical Methods Applicable to a Production Line with Stochastic Servers. In Algorithmic Methods in Probability, M.F. Neuts Editor, TIMS Studies in the Management Science, 7, 143-159.

Muth, E.J. 1979a. Modeling and Estimation of the Availability of Serial Production Lines. Research Report No. 79-4. University of Florida, Gainesville.

Muth, E.J. 1979b. The Reversibility Property of Production Lines. Management Science, 25, 152-158.

Muth, E.J. 1984. Stochastic Processes and Their Network Representations Associated with Production Line Queueing Model. European Journal of Operational Research, 15, 63-83.

Muth, E.J., and S. Yeralan, 1981. Effect of Buffer Size on Productivity of Work Stations that are Subject to Breakdown. Proc. The 20th IEEE Conference on Decision and Control, 643-648.

Neuts, M.F. 1981. Matrix-Geometric Solutions in Stochastic Models. The John Hopkins University Press, Baltimore, MD.

Okamura, K., and H. Yamashima. 1977. Analysis of the Effect of Buffer Storage Capacity in Transfer Line Systems. AIIE Transactions, 9, 127-135.

Rao, N.P. 1975a. On the Mean Production Rate of a Two-Stage Production System of the Tandem Type. International Journal of Production Research. 13, 207-217.

Rao, N.P. 1975b. Two-Stage Production Systems with Intermediate Storage. AIIE Transactions, 7, 414-421.

Rao, N.P. 1976. A Generalization of the "Bowl Phenomenon" in Series Production Systems. International Journal of Production Research, 14, 437-443.

Sevast'yanov, B.A. 1962. Influence of Storage Bin Capacity on the Average Standstill Time of a Production Line. Theory of Probability and Its Applications, 7, 429-438.

Shantikumar, J.G., and C.C. Tien. 1983. An Algorithmic Solution to Two-Stage Transfer Lines with Possible Scrapping of Units. Management Science, 29, 1069-1086.

Sheskin, T.J. 1976. Allocation of Interstage Storage Along an Automatic Production Line. AIIE Transactions, 8, 146-152.

Vladzievskii, A.P. 1952. Probabilistic Law of Operation and Internal Storage of Automatic Lines (in Russian). Automatika i Telemakhanika, 13, 227-281.

Wijngaard, J. 1979. The effect of Interstage Buffer Storage on the Output of Two Unreliable Production Units in Series, with Different Production Rates. AIIE Transactions, 11, 42-47.

Yeralan, S. 1983. Analysis of Serial Production Lines that are Subject to Breakdown. Ph.D. Dissertation, Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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